

Pointwise asymptotic behavior of modulated periodic reaction-diffusion waves

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Abstract

By working with the periodic resolvent kernel and Bloch-decomposition, we establish pointwise bounds for the Green function of the linearized equation associated with spatially periodic traveling waves of a system of reaction diffusion equations. With our linearized estimates together with a nonlinear iteration scheme developed by Johnson-Zumbrun, we obtain L^p -behavior ($p \geq 1$) of a nonlinear solution to a perturbation equation of a reaction-diffusion equation with respect to initial data in $L^1 \cap H^1$ recovering and slightly sharpening results obtained by Schneider using weighted energy and renormalization techniques. We obtain also pointwise nonlinear estimates with respect to two different initial perturbations $|u_0| \leq E_0 e^{-|x|^2/M}$ and $|u_0| \leq E_0(1 + |x|)^{-3/2}$, respectively, $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large, showing that behavior is that of a heat kernel. These pointwise bounds have not been obtained elsewhere, and do not appear to be accessible by previous techniques.

1 Introduction

In this paper, we obtain pointwise bounds for the Green function of the linearized equations associated with a spatially periodic traveling wave of a system of reaction diffusion equations, and use this to obtain pointwise bounds on decay and asymptotic behavior, sharpening bounds of [JZ2] and [S1, S2], of perturbations of a periodic traveling wave of a system of reaction diffusion equations. Suppose that $u(x, t) = \bar{u}(x - at)$ is a spatially periodic wave of a system of reaction diffusion equations of form $u_t = u_{xx} + f(u)$, where $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, $u \in \mathbb{R}^n$, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is sufficiently smooth: equivalently, $u(x, t) = \bar{u}(x)$ is a spatially periodic standing-wave solution of

$$(1.1) \quad u_t - au_x = u_{xx} + f(u).$$

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Throughout our analysis, we assume the existence of an X -periodic solution $\bar{u}(x)$ of (1.1). Without loss of generality, we assume that \bar{u} is 1-periodic, that is, $\bar{u}(x+1) = \bar{u}(x)$ for all $x \in \mathbb{R}$. A different pointwise Green function approach was carried out in [OZ] in the context of parabolic conservation laws by direct inverse Laplace transform computations not using the standard Bloch decomposition into periodic waves. In this paper we work from the Bloch representation and in the process we develop an interesting new formula for the high-frequency description of the resolvent of an operator with periodic boundary conditions on $[0, 1]$.

Linearizing (1.1) about a standing-wave solution $\bar{u}(x)$ gives the eigenvalue equation

$$(1.2) \quad \lambda v = Lv := (\partial_x^2 + a\partial_x + df(\bar{u}))v.$$

As coefficients of L are 1-periodic, Floquet theory implies that the L^2 spectrum is purely continuous and corresponds to the union of λ such that (1.2) admits a bounded eigenfunction of the form

$$(1.3) \quad v(x) = e^{i\xi x}w(x), \quad \xi \in \mathbb{R}$$

where $w(x+1) = w(x)$, that is, the eigenvalues of the family of associated Floquet, or Bloch, operators

$$(1.4) \quad L_\xi := e^{-i\xi x}Le^{i\xi x} = (\partial_x + i\xi)^2 + a(\partial_x + i\xi) + df(\bar{u}), \quad \text{for } \xi \in [-\pi, \pi),$$

considered as acting on L^2 periodic functions on $[0, 1]$.

Recall that any function $g \in L^2(\mathbb{R})$ admits an inverse Bloch-Fourier representation

$$(1.5) \quad g(x) = \int_{-\pi}^{\pi} e^{i\xi x} \hat{g}(\xi, x) d\xi.$$

where $\hat{g}(\xi, x) = \sum_{j \in \mathbb{Z}} e^{j2\pi i x} \hat{g}(\xi + j2\pi i)$ is a 1-periodic functions of x , and $\hat{g}(\cdot)$ denotes the Fourier transform of g with respect to x . Indeed, using the Fourier transform we have

$$(1.6) \quad g(x) = \int_{-\infty}^{\infty} e^{i\xi x} \hat{g}(\xi) d\xi = \sum_{j \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{i(\xi + j2\pi i)x} \hat{g}(\xi + j2\pi i) d\xi = \int_{-\pi}^{\pi} e^{i\xi x} \hat{g}(\xi, x) d\xi.$$

Since $L(e^{i\xi x}f) = e^{i\xi x}(L_\xi f)$ for f periodic, the Bloch-Fourier transform diagonalizes the periodic-coefficient operator L , yielding the inverse Bloch-Fourier transform representation

$$(1.7) \quad e^{Lt}g(x) = \int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t} \hat{g}(\xi, x) d\xi.$$

By the translation invariance of (1.1), the function $\bar{u}'(x)$ is a 1-periodic solution of the differential equation $L_0 v = 0$. Hence, it follows that $\lambda = 0$ is an eigenvalue of the Bloch operator L_0 . Define following [S1, S2, JZ2] the *diffusive spectral stability* conditions:

(D1) $\lambda = 0$ is a simple eigenvalue of L_0 .

(D2) $\operatorname{Re}\sigma(L_\xi) \leq -\theta|\xi|^2$, $\theta > 0$, for all real ξ with $|\xi|$ sufficiently small.

Assumption (D1) corresponds to transversality of \bar{u} as a solution of the associated traveling-wave ODE, while assumption (D2) corresponds to “dissipativity” of the large-time behavior of the linearized system; see [S1, S2, JZ2].

Remark 1.1 ([JZ2]). *By standard spectral perturbation theory [K], (D1) implies that the eigenvalue $\lambda(\xi)$ bifurcating from $\lambda = 0$ at $\xi = 0$ is analytic at $\xi = 0$, with $\lambda(\xi) = \lambda_1\xi + \lambda_2\xi^2 + O(|\xi|^3)$, from which we find from the necessary stability condition $\operatorname{Re}\lambda(\xi) \leq 0$ that $\operatorname{Re}\lambda_1 = 0$ and $\operatorname{Re}\lambda_2 \leq 0$. Assumption (D2) thus amounts to the nondegeneracy condition $\operatorname{Re}\lambda_2 \neq 0$ together with the strict stability condition $\operatorname{Re}\sigma(L_\xi) < 0$ for $\xi \neq 0$.*

Rewriting the eigenvalue equation (1.2) as a first-order system

$$(1.8) \quad V' = \mathbb{A}(\lambda, x)V,$$

where

$$V = \begin{pmatrix} v \\ v' \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} 0 & I \\ \lambda I - df(\bar{u}) & -aI \end{pmatrix},$$

denote by $\mathcal{F}^{y \rightarrow x} \in \mathbb{C}^{2n \times 2n}$ the solution operator of (1.8), defined by $\mathcal{F}^{y \rightarrow y} = I$, $\partial_x \mathcal{F} = \mathbb{A}\mathcal{F}$. That is, $\mathcal{F}^{y \rightarrow x} = \Phi(x)\Phi(y)^{-1}$, for any fundamental matrix solution Φ of the (1.8).

By the definition of Bloch operators (1.4), for each $\xi \in [-\pi, \pi)$, we have a second-order eigenvalue equation

$$(1.9) \quad \lambda u = L_\xi u = u'' - A_\xi u' - C_\xi u,$$

where $A_\xi = -(a + 2i\xi)I \in \mathbb{C}^{n \times n}$ a constant matrix and $C_\xi(x) = -df(\bar{u}) - (ia\xi - \xi^2)I \in \mathbb{C}^{n \times n}$ a matrix depending on x , and $u \in \mathbb{C}^n$ is a vector.

Rewriting (1.9) as a first-order system

$$(1.10) \quad U' = \mathbb{A}_\xi(x, \lambda)U,$$

where

$$(1.11) \quad U = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad \mathbb{A}_\xi = \begin{pmatrix} 0 & I \\ \lambda I + C_\xi & A_\xi \end{pmatrix},$$

similarly, denote by $\mathcal{F}_\xi^{y \rightarrow x} \in \mathbb{C}^{2n \times 2n}$ the solution operator of (1.10), defined by $\mathcal{F}_\xi^{y \rightarrow y} = I$, $\partial_x \mathcal{F}_\xi = \mathbb{A}_\xi \mathcal{F}_\xi$. That is, $\mathcal{F}_\xi^{y \rightarrow x} = \Phi_\xi(x)\Phi_\xi(y)^{-1}$, for any fundamental matrix solution Φ_ξ of the (1.10).

1.1 Main result

With these preparations, we now state our two main results.

Theorem 1.2. *The Green function $G(x, t; y)$ for equation (1.2) satisfies the estimates:*

$$(1.12) \quad \begin{aligned} G(x, t; y) &= \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} q(x, 0) \tilde{q}(y, 0) + O((1+t)^{-1} + t^{-\frac{1}{2}} e^{-\eta t}) e^{-\frac{|x-y-at|^2}{Mt}}, \\ G_y(x, t; y) &= \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} q(x, 0) \tilde{q}(y, 0) + O(t^{-1}) e^{-\frac{|x-y-at|^2}{Mt}}, \end{aligned}$$

uniformly on $t \geq 0$, for some sufficiently large constants $M > 0$ and $\eta > 0$, where q and \tilde{q} are the periodic right and left eigenfunctions of L_0 , respectively, at $\lambda = 0$. In particular $q(x, 0) = \bar{u}'(x)$.

Theorem 1.3. *Define the nonlinear perturbation $u := \tilde{u} - \bar{u}$, where \tilde{u} satisfies (1.1). Then the asymptotic behavior of u with respect to three kinds of initial data (denoted by u_0):*

(1) $|u_0(x)|_{L^1 \cap H^1}, |xu_0|_{L^1} < E_0$, sufficiently small

(2) $|u_0(x)| < E_0 e^{-\frac{|x|^2}{M}}$, $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large

(3) $|u_0(x)| < E_0(1 + |x|)^{-r}$, $E_0 > 0$ sufficiently small and $r > 2$

converges to a heat kernel with the following estimates, respectively

(a) $|u(x, t) - \bar{U}_* \bar{u}' \bar{k}(x, t)|_{L^p(x)} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+\ln(1+t))$, for $1 \leq p \leq \infty$

(b) $|u(x, t) - \bar{U}_* \bar{u}' \bar{k}(x, t)| \leq CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}} (1+\ln(1+t))$

(c) $|u(x, t) - \bar{U}_* \bar{u}' \bar{k}(x, t)| \leq CE_0 \left[(1+t)^{-\frac{1}{2}} (1+|x-at|+\sqrt{t})^{-r+1} + (1+t)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}} (1+\ln(1+t)) \right],$

for $\bar{k}(x, t) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-at|^2}{4bt}}$, $M'' > M$ and $C > 0$ sufficiently large and some constant \bar{U}_* (defined in Section 6).

Remark 1.4. *The 3 parts of Theorem 1.3 is established in Theorem 6.7, 6.23 and 7.13, respectively.*

Remark 1.5. *The initial condition $|u_0|_{L^1 \cap H^1}, |xu_0|_{L^1}$ sufficiently small is compared with Schneider's [S2] initial assumption. By Fourier transform, we can roughly consider $|(1+|x|^2)u_0|_{H^2}$ as Schneider's initial condition with weight $(1+|x|^2)$ (See Schneider [S2], pp690-691). This implies that our initial data roughly satisfies $|u_0| \lesssim |x|^{-2}$ whereas Schneider's initial data roughly satisfies $|u_0| \lesssim |x|^{-\frac{5}{2}}$. Our L^p bounds on asymptotic behavior for all $p \geq 1$ are also compared with Schneider's L^∞ bound. In particular, our L^∞ bound $t^{-1} \ln(1+t)$ is roughly equivalent to but slightly sharper than Schneider's L^∞ bound $t^{-1+\varepsilon}$ for $\varepsilon > 0$. Though Schneider does not state L^p bounds, his renormalized $H^2(2)$ bounds (see Thm. 15, [S2]) by a simple scaling argument yield L^p bounds $\sim t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}+\eta}$ for any $\eta > 0$, for all $p \geq 1$, again roughly equivalent to but slightly less sharp than ours.*

1.2 Discussion and open problems

Pointwise Green function bounds have been obtained by Oh and Zumbrun previously for systems of conservation laws, by somewhat different methods, without use of the Bloch

representation. Those methods would work here as well; however, we find the present method proceeding from the Bloch transform both more direct and more connected to other literature in the area; in particular, it makes a direct connection between the Oh-Zumbrun analysis and other works, filling in the previously missing link of pointwise Green function bounds for periodic-coefficient operators on a bounded periodic domain, a topic that seems of interest in its own right. In addition, the analysis has a flavor of explicit, spatial domain computation that illuminates the arguments of Schneider, Johnson-Zumbrun, and others by weighted energy estimates, Hausdorff-Young inequality, and other frequency domain techniques.

A novel aspect of the present work is to obtain pointwise bounds also on the nonlinear solution, and thereby sharp L^p bounds for all $1 \leq p \leq \infty$. Schneider's weighted H^2 estimates, obtained by renormalization techniques, yield L^p bounds for $1 \leq p \leq \infty$ of $(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}+\eta}$ for any $\eta > 0$, just slightly weaker than ours; however, the estimates of Johnson-Zumbrun, obtained by Hausdorff-Young's inequality appear limited to $2 \leq p \leq \infty$. The more detailed pointwise bounds we obtain here do not seem to be accessible by either of these previous two techniques.

An important advantage of our approach over the renormalization techniques used by Schneider and others, is that, being based rather on the nonlinear tracking scheme of Johnson-Zumbrun, it should apply in principle also to situations, such as periodic solutions of conservation laws like the Kuramoto-Sivashinsky equations and others, for which the asymptotic behavior consists of multiple signals convecting with distinct speeds; see for example the analysis of [JZ1, JZN, JNRZ]. By contrast, renormalization techniques appear limited to situations of a single signal. The extension of our results to the conservation law case is an interesting open problem.

Finally, we mention that the techniques used here extend to general quasilinear parabolic or even mixed, partially parabolic problems, so that our analysis could in principle extend to these more general settings; see, for example, the related analyses in [HZ, RZ, JZN]. This would be another very interesting direction to carry out.

2 The resolvent kernel

In this section, we develop an interesting formula for the resolvent kernel on the whole line and for periodic boundary conditions on $[0, 1]$ using solution operators and projections. This formula is motivated by the constant-coefficient scalar case (see Section 5). We will use this formula to find a high-frequency description of the resolvent for periodic boundary condition $[0, 1]$ in Section 3.

For λ in the resolvent set of L , we denote by $G_\lambda(x, y)$ the resolvent kernel defined by

$$(2.1) \quad (L - \lambda I)G_\lambda(\cdot, y) := \delta_y \cdot I,$$

δ_y denoting the Dirac delta distribution centered at y , or equivalently

$$(2.2) \quad (L - \lambda I)^{-1}f(x) = \int G_\lambda(x, y)f(y)dy.$$

For each $\xi \in [-\pi, \pi)$ and for λ in the resolvent set of L_ξ , we denote by $\mathcal{G}_{\xi, \lambda}(x, y)$ and $G_{\xi, \lambda}(x, y)$ the resolvent kernels of L_ξ on the whole line and on $[0, 1]$ with periodic boundary conditions, respectively.

Remark 2.1. *The spectrum of each L_ξ may alternatively be characterized as the zero set for fixed ξ of the periodic Evans function introduced by Gardner in [G1] and [G2],*

$$D(\lambda, \xi) = \det(\Psi(\lambda) - e^{i\xi}I),$$

where Ψ is the monodromy matrix of (1.8), and $D(\lambda, \xi)$ is analytic in each argument λ and ξ ; likewise, the spectrum of L may be described as the set of all λ such that $D(\lambda, \xi)$ vanished for some real ξ . So if λ is in the resolvent set of L , then

$$(2.3) \quad \det(\Psi(\lambda) - e^{i\xi}I) \neq 0 \quad \text{for all } \xi \in \mathbb{R},$$

that is, $\mathcal{F}^{y \rightarrow y+1} - e^{i\xi}I$ is invertible for all $\xi \in \mathbb{R}$. Using decomposition

$$(2.4) \quad \mathcal{F}^{y \rightarrow y+1} = e^{i\xi} \begin{pmatrix} I & 0 \\ i\xi I & I \end{pmatrix} \mathcal{F}_\xi^{y \rightarrow y+1} \begin{pmatrix} I & 0 \\ i\xi I & I \end{pmatrix}^{-1},$$

$I - \mathcal{F}_\xi^{y \rightarrow y+1}$ is invertible for all $\xi \in \mathbb{R}$. Also (2.3) implies the existence of Π^\pm and Π_ξ^\pm because $\Psi(\lambda)$ does not have eigenvalue of norm 1.

2.1 The whole line case

Lemma 2.2. *For all $\xi \in [-\pi, \pi)$, the whole line kernel (See the definition above) satisfies*

$$(2.5) \quad \begin{pmatrix} \mathcal{G}_{\xi, \lambda} \\ \mathcal{G}'_{\xi, \lambda} \end{pmatrix} (x, y) = \begin{cases} \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^+(y) \begin{pmatrix} 0 \\ I \end{pmatrix}, & x > y, \\ -\mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^-(y) \begin{pmatrix} 0 \\ I \end{pmatrix}, & x \leq y, \end{cases}$$

where Π_ξ^\pm are projections onto the manifolds of solutions decaying as $x \rightarrow \pm\infty$.

Proof. We must only check the jump condition $\left[\begin{pmatrix} \mathcal{G}_{\xi, \lambda} \\ \mathcal{G}'_{\xi, \lambda} \end{pmatrix} \right]_{|y} = \begin{pmatrix} 0 \\ I \end{pmatrix}$, which follows from $\mathcal{F}_\xi^{y \rightarrow y} = I$ and $\Pi_\xi^+ + \Pi_\xi^- = I$, and the fact that $\mathcal{G}_{\xi, \lambda}(x, y) \rightarrow 0$ as $x \rightarrow \pm\infty$, which is clear by inspection. \square

2.2 The periodic case

Lemma 2.3. *For λ in the resolvent set of L and all $\xi \in [-\pi, \pi)$, the periodic kernel satisfies*

$$(2.6) \quad \begin{pmatrix} G_{\xi, \lambda} \\ G'_{\xi, \lambda} \end{pmatrix} (x, y) = \begin{cases} \mathcal{F}_\xi^{y \rightarrow x} M_\xi^+(y) \begin{pmatrix} 0 \\ I \end{pmatrix}, & x > y, \\ -\mathcal{F}_\xi^{y \rightarrow x} M_\xi^-(y) \begin{pmatrix} 0 \\ I \end{pmatrix}, & x \leq y, \end{cases}$$

where $M_{\xi}^{+}(y) = (I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1}$ and $M_{\xi}^{-}(y) = -(I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} \mathcal{F}_{\xi}^{y \rightarrow y+1}$.

(Note: Remark 2.1 implies the existence of M_{ξ}^{+} and M_{ξ}^{-} .)

Proof. We must check the jump condition $\left[\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix} \right]_y = \begin{pmatrix} 0 \\ I \end{pmatrix}$, which follows from $\mathcal{F}_{\xi}^{y \rightarrow y} = I$ and $M_{\xi}^{+} + M_{\xi}^{-} = I$, and the periodicity, $\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix}(y, 1) = \begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix}(y, 0)$. By the periodicity of the solution operator, $\mathcal{F}_{\xi}^{0 \rightarrow y} \mathcal{F}_{\xi}^{y \rightarrow 1} = \mathcal{F}_{\xi}^{1 \rightarrow y+1} \mathcal{F}_{\xi}^{y \rightarrow 1} = \mathcal{F}_{\xi}^{y \rightarrow y+1}$. By a direct computation, we obtain $\mathcal{F}_{\xi}^{y \rightarrow 1} (I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} = \mathcal{F}_{\xi}^{y \rightarrow 0} (I - \mathcal{F}_{\xi}^{y \rightarrow y+X})^{-1} \mathcal{F}_{\xi}^{y \rightarrow y+1}$ which gives us $\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix}(y, 1) = \begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix}(y, 0)$. \square

3 Pointwise bounds on $G_{\xi,\lambda}$ for $|\lambda| > R$, R sufficiently large

For the proof of lemma 3.1, we follow the proof of high frequency bounds which come from Zumbrun-Howard([ZH]).

Lemma 3.1. *For each $|\xi| \leq \pi$ and for sufficiently large $|\lambda|$,*

$$(3.1) \quad \begin{aligned} \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{+}(y) &= e^{-\beta^{-1/2} |\lambda|^{1/2} |(x-y)|} N_1 O(1) N_2, \quad \text{for } x > y, \\ \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{-}(y) &= e^{-\beta^{-1/2} |\lambda|^{1/2} |(y-x)|} N_1 O(1) N_2, \quad \text{for } x \leq y, \end{aligned}$$

where $N_1 = \begin{pmatrix} |\lambda|^{-1/2} I & 0 \\ 0 & I \end{pmatrix}$, $N_2 = \begin{pmatrix} |\lambda|^{1/2} I & 0 \\ 0 & I \end{pmatrix}$ and Π_{ξ}^{\pm} projections onto the manifolds of solutions decaying as $x \rightarrow \pm\infty$, and here $\beta^{-1/2} \sim \min_{\{\lambda: \operatorname{Re} \lambda \geq \eta_1 - \eta_2 | \operatorname{Im} \lambda| \}} \operatorname{Re}(\sqrt{\lambda/|\lambda|})$.¹

Proof. Setting $\bar{x} = |\lambda|^{1/2} x$, $\bar{\lambda} = \lambda/|\lambda|$, $\bar{u}(\bar{x}) = u(\bar{x}/|\lambda|^{1/2})$, $\bar{C}(\bar{x}) = C(\bar{x}/|\lambda|^{1/2})$, in (1.9), we obtain

$$(3.2) \quad \bar{u}'' = \bar{\lambda} \bar{u} + |\lambda|^{-1/2} A_{\xi} \bar{u}' + |\lambda|^{-1} \bar{C}_{\xi} \bar{u},$$

or

$$(3.3) \quad \bar{U}' = \bar{\mathbb{A}} \bar{U} + \Theta_{\xi} \bar{U},$$

where $\bar{U} = \begin{pmatrix} \bar{u} \\ \bar{u}' \end{pmatrix}$, $\bar{\mathbb{A}} = \begin{pmatrix} 0 & I \\ \bar{\lambda} I & 0 \end{pmatrix}$, $\Theta_{\xi} = \begin{pmatrix} 0 & 0 \\ |\lambda|^{-1} \bar{C}_{\xi} & |\lambda|^{-1/2} A_{\xi} \end{pmatrix}$ and $|\bar{\lambda}| = 1$. Denote by $\bar{\mathcal{F}}_{\xi}^{\bar{y} \rightarrow \bar{x}}$ the solution operator of (3.3) and by $\bar{\Pi}_{\xi}^{\pm}$ projections onto the manifolds of solutions decaying as $x \rightarrow \pm\infty$.

It is easily computed that the eigenvalues of $\bar{\mathbb{A}}$ are $\mp \sqrt{\bar{\lambda}}$ and

$$(3.4) \quad \operatorname{Re} \sqrt{\bar{\lambda}} > \beta^{-1/2}$$

¹Here and elsewhere in this section, $O(1)$ is matrix-valued, denoting a matrix with bounded coefficients.

for all $\lambda \in \{Re\lambda \geq \eta_1 - \eta_2 |Im\lambda|\}$ for some $\beta > 0$ and $\eta_1, \eta_2 > 0$, hence the stable and unstable subspaces of each \mathbb{A} are both of dimension n , and separated by a spectral gap of more than 2β . Let $P = \begin{pmatrix} P_+ \\ P_- \end{pmatrix}$, where rows of P_{\pm} are left eigenvectors corresponding $\mp\sqrt{\lambda}$, respectively.

Introducing new coordinates $w_{\pm} = P_{\pm}\bar{U}$ and using $P\bar{\mathbb{A}}P^{-1} = \begin{pmatrix} -\sqrt{\lambda}I & 0 \\ 0 & \sqrt{\lambda}I \end{pmatrix}$, we obtain a block diagonal system

$$(3.5) \quad \begin{pmatrix} w_+ \\ w_- \end{pmatrix}' = \begin{pmatrix} -\sqrt{\lambda}I & 0 \\ 0 & \sqrt{\lambda}I \end{pmatrix} \begin{pmatrix} w_+ \\ w_- \end{pmatrix} + \bar{\Theta}_{\xi} \begin{pmatrix} w_+ \\ w_- \end{pmatrix},$$

where

$$(3.6) \quad \begin{aligned} \bar{\Theta}_{\xi} &= P\Theta_{\xi}P^{-1} \\ &= \frac{1}{2} \begin{pmatrix} I & -\sqrt{\lambda}^{-1} \\ I & \sqrt{\lambda}^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ |\lambda^{-1}|\bar{C}_{\xi} & |\lambda^{-\frac{1}{2}}|A_{\xi} \end{pmatrix} \begin{pmatrix} I & I \\ -\sqrt{\lambda} & \sqrt{\lambda} \end{pmatrix} \\ &= \frac{1}{2}|\lambda^{-\frac{1}{2}}| \begin{pmatrix} -\lambda^{-\frac{1}{2}}\bar{C}_{\xi} + A_{\xi} & -\lambda^{-\frac{1}{2}}\bar{C}_{\xi} - A_{\xi} \\ \lambda^{-\frac{1}{2}}\bar{C}_{\xi} - A_{\xi} & \lambda^{-\frac{1}{2}}\bar{C}_{\xi} + A_{\xi} \end{pmatrix} \\ &= |\lambda^{-\frac{1}{2}}| \begin{pmatrix} \theta_{\xi_{11}} & \theta_{\xi_{12}} \\ \theta_{\xi_{21}} & \theta_{\xi_{22}} \end{pmatrix}. \end{aligned}$$

Since $|\lambda^{-\frac{1}{2}}|$ is sufficiently small for $|\lambda|$ sufficiently large, by using the tracking lemma (see [MaZ], p20), there is a unique linear transformation

$$(3.7) \quad S = \begin{pmatrix} I & \Phi_+ \\ \Phi_- & I \end{pmatrix} \quad \text{with} \quad |\Phi_{\pm}| \leq |\lambda^{-\frac{1}{2}}|$$

so that new coordinates $w_{\pm} = Sz_{\pm}$ generate an exact block diagonal system

$$(3.8) \quad \begin{pmatrix} z_+ \\ z_- \end{pmatrix}' = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix},$$

where $A_+ = -\sqrt{\lambda}I + |\lambda^{-\frac{1}{2}}|(\theta_{\xi_{11}} + \theta_{\xi_{12}}\Phi_-)$, and $A_- = \sqrt{\lambda}I + |\lambda^{-\frac{1}{2}}|(\theta_{\xi_{21}}\Phi_+ + \theta_{\xi_{22}})$.

For any $|\xi| \leq \pi$ and for $i, j = 1, 2$, $|\theta_{\xi_{ij}}| = O(|\lambda^{-\frac{1}{2}}|(C - (ia\xi + \xi^2)I) + (a - 2i\xi)I|)$, and so $\theta_{\xi_{11}} + \theta_{\xi_{12}}\Phi_- = O(1) = \theta_{\xi_{21}}\Phi_+ + \theta_{\xi_{22}}$ for sufficiently large $|\lambda|$.

Now we have $z'_+ = (-\sqrt{\lambda}I + O(|\lambda^{-\frac{1}{2}}|))z_+$ and $z'_- = (\sqrt{\lambda}I + O(|\lambda^{-\frac{1}{2}}|))z_-$. From this we obtain the energy estimate,

$$\begin{aligned} \langle z_{\pm}, z_{\pm} \rangle' &= \langle z_{\pm}, \mp Re\sqrt{\lambda}I z_{\pm} \rangle + O(|\lambda^{-\frac{1}{2}}|)\langle z_{\pm}, z_{\pm} \rangle \\ &\leq (\mp\beta^{-1/2} + O(|\lambda^{-\frac{1}{2}}|))\langle z_{\pm}, z_{\pm} \rangle. \end{aligned}$$

So we find that

$$(|z_{\pm}|^2)' \leq (\mp\beta^{-1/2} + O(|\lambda^{-\frac{1}{2}}|))|z_{\pm}|^2,$$

hence

$$(3.9) \quad \begin{aligned} \frac{|z_+(\bar{x})|}{|z_+(\bar{y})|} &\leq e^{-\beta^{-1/2}(\bar{x}-\bar{y})}, \quad \text{for } \bar{x} > \bar{y}, \\ \frac{|z_-(\bar{x})|}{|z_-(\bar{y})|} &\leq e^{-\beta^{-1/2}(\bar{y}-\bar{x})}, \quad \text{for } \bar{x} \leq \bar{y}, \end{aligned}$$

provided $|\lambda|$ is sufficiently large. Since $|S| = O(1 + |\lambda^{-\frac{1}{2}}|)$ and $|P| = O(1)$, translating the bound (3.9) back to (3.3), we obtain for any $|\xi| \leq \pi$,

$$(3.10) \quad \begin{aligned} \bar{\mathcal{F}}_{\xi}^{\bar{y} \rightarrow \bar{x}} \bar{\Pi}_{\xi}^{+}(\bar{y}) &= O(1)e^{-\beta^{-1/2}(\bar{x}-\bar{y})}, \quad \text{for } \bar{x} > \bar{y}, \\ \bar{\mathcal{F}}_{\xi}^{\bar{y} \rightarrow \bar{x}} \bar{\Pi}_{\xi}^{-}(\bar{y}) &= O(1)e^{-\beta^{-1/2}(\bar{y}-\bar{x})}, \quad \text{for } \bar{x} \leq \bar{y}. \end{aligned}$$

provided $|\lambda|$ is sufficiently large.

The operators $\mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{\pm}(y)$ are evidently related to the corresponding operators $\bar{\mathcal{F}}_{\xi}^{\bar{y} \rightarrow \bar{x}} \bar{\Pi}_{\xi}^{\pm}(y)$ for the rescaled system by the scaling transformation

$$(3.11) \quad \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{\pm}(y) = \begin{pmatrix} |\lambda^{-1/2}|I & 0 \\ 0 & I \end{pmatrix} \bar{\mathcal{F}}_{\xi}^{|\lambda^{1/2}|y \rightarrow |\lambda^{1/2}|x} \bar{\Pi}_{\xi}^{\pm}(y) \begin{pmatrix} |\lambda^{1/2}|I & 0 \\ 0 & I \end{pmatrix}.$$

From (3.10) and $\bar{\Pi}_{\xi}^{\pm}(y) = O(1)$, we thus have

$$(3.12) \quad \begin{aligned} \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{+}(y) &= e^{-\beta^{-1/2}|\lambda^{1/2}|(x-y)} \begin{pmatrix} |\lambda^{-1/2}|I & 0 \\ 0 & I \end{pmatrix} O(1) \begin{pmatrix} |\lambda^{1/2}|I & 0 \\ 0 & I \end{pmatrix}, \quad \text{for } x > y, \\ \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{-}(y) &= e^{-\beta^{-1/2}|\lambda^{1/2}|(y-x)} \begin{pmatrix} |\lambda^{-1/2}|I & 0 \\ 0 & I \end{pmatrix} O(1) \begin{pmatrix} |\lambda^{1/2}|I & 0 \\ 0 & I \end{pmatrix}, \quad \text{for } x \leq y, \end{aligned}$$

provided $|\lambda|$ is sufficiently large. \square

Proposition 3.2. *For any $|\xi| \leq \pi$ and any $x \in [0, 1]$,*

$$(3.13) \quad \begin{aligned} |G_{\xi,\lambda}(x, y)| &\leq C|\lambda^{-1/2}|(e^{-\beta^{-1/2}|\lambda^{1/2}||x-y|} + e^{-\beta^{-1/2}|\lambda^{1/2}|(1-|x-y|)}) \\ |(\partial/\partial_y)G_{\xi,\lambda}(x, y)| &\leq C(e^{-\beta^{-1/2}|\lambda^{1/2}||x-y|} + e^{-\beta^{-1/2}|\lambda^{1/2}|(1-|x-y|)}) \end{aligned}$$

provided $|\lambda|$ is sufficiently large and $C > 0$, that is, $|G_{\xi,\lambda}|$ is uniformly bounded as $|\lambda| \rightarrow \infty$.

Proof. We note that, by the periodicity of the resolvent kernel,

$$(3.14) \quad \mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{\pm}(y) = \Pi_{\xi}^{\pm}(y+1) \mathcal{F}_{\xi}^{y \rightarrow y+1} = \Pi_{\xi}^{\pm}(y) \mathcal{F}_{\xi}^{y \rightarrow y+1},$$

which implies

$$(3.15) \quad \Pi_{\xi}^{\pm}(y)(I - \mathcal{F}_{\xi}^{y \rightarrow y+1})(I - \Pi_{\xi}^{\pm}(y) \mathcal{F}_{\xi}^{y \rightarrow y+1}) = (I - \Pi_{\xi}^{\pm}(y) \mathcal{F}_{\xi}^{y \rightarrow y+1}) \Pi_{\xi}^{\pm}(y)(I - \mathcal{F}_{\xi}^{y \rightarrow y+1}).$$

Now, recall the resolvent kernel for the periodic case as

$$\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix} (x, y) = \begin{cases} \mathcal{F}_{\xi}^{y \rightarrow x} M_{\xi}^{+}(y) \begin{pmatrix} 0 \\ I \end{pmatrix}, & x > y, \\ -\mathcal{F}_{\xi}^{y \rightarrow x} M_{\xi}^{-}(y) \begin{pmatrix} 0 \\ I \end{pmatrix}, & x \leq y, \end{cases}$$

where $M_{\xi}^{+}(y) = (I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1}$ and $M_{\xi}^{-}(y) = -(I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} \mathcal{F}_{\xi}^{y \rightarrow y+1}$.

Let's consider the case of $x > y$ first. Since $\Pi_{\xi}^{+} + \Pi_{\xi}^{-} = I$,

$$\mathcal{F}_{\xi}^{y \rightarrow x} M_{\xi}^{+}(y) = \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{+}(y) M_{\xi}^{+}(y) + \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{-}(y) M_{\xi}^{+}(y).$$

From (3.12) and (3.15) and recalling that $N_1 = \begin{pmatrix} |\lambda^{-1/2}|I & 0 \\ 0 & I \end{pmatrix}$, $N_2 = \begin{pmatrix} |\lambda^{1/2}|I & 0 \\ 0 & I \end{pmatrix}$, we have for $x > y$,

$$\begin{aligned} & \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{+}(y) M_{\xi}^{+}(y) \\ &= \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{+}(y) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{+}(y)) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{+}(y))^{-1} (I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} \\ &= \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{+}(y) \Pi_{\xi}^{+}(y) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1}) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{+}(y))^{-1} (I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} \\ (3.16) \quad &= \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{+}(y) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{+}(y))^{-1} \Pi_{\xi}^{+}(y) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1}) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} \\ &= \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{+}(y) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{+}(y))^{-1} \Pi_{\xi}^{+}(y) \\ &= e^{-\beta^{-1/2} |\lambda^{1/2}| (x-y)} N_1 O(1) N_2, \end{aligned}$$

where we have used the fact that $\mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{+}(y)$ is decaying for $|\lambda|$ sufficiently large. Similarly, we have

$$\begin{aligned} & \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{-}(y) M_{\xi}^{+}(y) = \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{-}(y) (I - \mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{-}(y))^{-1} \Pi_{\xi}^{-}(y) \\ & \approx \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{-}(y) (\mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{-}(y))^{-1} \Pi_{\xi}^{-}(y) \\ (3.17) \quad &= \mathcal{F}_{\xi}^{y \rightarrow x} \Pi_{\xi}^{-}(y) \Pi_{\xi}^{-}(y) \mathcal{F}_{\xi}^{y+1 \rightarrow y} \\ &= \mathcal{F}_{\xi}^{y+1 \rightarrow x} \Pi_{\xi}^{-}(y) \\ &= e^{-\beta^{-1/2} |\lambda^{1/2}| (y+1-x)} N_1 O(1) N_2, \end{aligned}$$

here, the above approximation is from the fact that $\mathcal{F}_{\xi}^{y \rightarrow y+1} \Pi_{\xi}^{-}(y)$ is growing for $|\lambda|$ sufficiently large.

So, for $x > y$,

$$\begin{aligned} (3.18) \quad & \begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix} (x, y) = (e^{-\beta^{-1/2} |\lambda^{1/2}| (x-y)} N_1 O(1) N_2 + e^{-\beta^{-1/2} |\lambda^{1/2}| (y+1-x)} N_1 O(1) N_2) \begin{pmatrix} 0 \\ I \end{pmatrix} \\ &= (e^{-\beta^{-1/2} |\lambda^{1/2}| (x-y)} + e^{-\beta^{-1/2} |\lambda^{1/2}| (y+1-x)}) \begin{pmatrix} O(|\lambda^{-1/2}|)I \\ O(1)I \end{pmatrix}. \end{aligned}$$

Now, we consider the case of $x \leq y$. From (3.15) and the calculation of (3.16), we have for $x \leq y$,

$$\begin{aligned}
 \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^+(y) M_\xi^-(y) &= \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^+(y) (I - \mathcal{F}_\xi^{y \rightarrow y+1})^{-1} \mathcal{F}_\xi^{y \rightarrow y+1} \\
 &= \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^+(y) (I - \mathcal{F}_\xi^{y \rightarrow y+1} \Pi_\xi^+(y))^{-1} \Pi_\xi^+(y) \mathcal{F}_\xi^{y \rightarrow y+1} \\
 &= \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^+(y) \mathcal{F}_\xi^{y \rightarrow y+1} (I - \mathcal{F}_\xi^{y \rightarrow y+1} \Pi_\xi^+(y))^{-1} \\
 &= \mathcal{F}_\xi^{y \rightarrow x} \mathcal{F}_\xi^{y \rightarrow y+1} \Pi_\xi^+(y) (I - \mathcal{F}_\xi^{y \rightarrow y+1} \Pi_\xi^+(y))^{-1} \\
 &= \mathcal{F}_\xi^{y+1 \rightarrow x+1} \mathcal{F}_\xi^{y \rightarrow y+1} \Pi_\xi^+(y) (I - \Pi_\xi^+(y) \mathcal{F}_\xi^{y \rightarrow y+1})^{-1} \\
 &= \mathcal{F}_\xi^{y \rightarrow x+1} \Pi_\xi^+(y) (I - \Pi_\xi^+(y) \mathcal{F}_\xi^{y \rightarrow y+1})^{-1} \\
 &= e^{-\beta^{-1/2} |\lambda^{1/2}| (x+1-y)} N_1 O(1) N_2.
 \end{aligned}
 \tag{3.19}$$

Similarly, we have

$$\begin{aligned}
 \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^-(y) M_\xi^-(y) &= \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^-(y) (I - \mathcal{F}_\xi^{y \rightarrow y+1})^{-1} \mathcal{F}_\xi^{y \rightarrow y+1} \\
 &= \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^-(y) (I - \mathcal{F}_\xi^{y \rightarrow y+1} \Pi_\xi^-(y))^{-1} \Pi_\xi^-(y) \mathcal{F}_\xi^{y \rightarrow y+1} \\
 &\approx \mathcal{F}_\xi^{y \rightarrow x} \Pi_\xi^-(y) (\mathcal{F}_\xi^{y \rightarrow y+1} \Pi_\xi^-(y))^{-1} \Pi_\xi^-(y) \mathcal{F}_\xi^{y \rightarrow y+1} \\
 &= \Pi_\xi^-(x) \mathcal{F}_\xi^{y \rightarrow x} \\
 &= e^{-\beta^{-1/2} |\lambda^{1/2}| (y-x)} N_1 O(1) N_2.
 \end{aligned}
 \tag{3.20}$$

So, for $x \leq y$,

$$\begin{aligned}
 \begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix} (x, y) &= (e^{-\beta^{-1/2} |\lambda^{1/2}| (x+1-y)} N_1 O(1) N_2 + e^{-\beta^{-1/2} |\lambda^{1/2}| (y-x)} N_1 O(1) N_2) \begin{pmatrix} 0 \\ I \end{pmatrix} \\
 &= (e^{-\beta^{-1/2} |\lambda^{1/2}| (x+1-y)} + e^{-\beta^{-1/2} |\lambda^{1/2}| (y-x)}) \begin{pmatrix} O(|\lambda^{-1/2}|) I \\ O(1) I \end{pmatrix}.
 \end{aligned}
 \tag{3.21}$$

This completes the proof of the proposition □

Remark 3.3. We can express (3.13) as

$$G_{\xi,\lambda}(x, y) = O(|\lambda^{-1/2}|) (e^{-\beta^{-1/2} |\lambda^{1/2}| \min|x-y_i|}),
 \tag{3.22}$$

where $y_j = y + j$.

Remark 3.4. The aliasing between y , $y - 1$ and $y + 1$ indicates why the periodic resolvent formula possesses always a “ $y < x$ ” type piece even when $y > x$. This comes from the influence of $y - 1$.

Remark 3.5. The periodic resolvent kernel $G_{\xi,\lambda}$ may also be obtained in indirect fashion from the whole-line version $\mathcal{G}_{\xi,\lambda}$ by the method of images

$$[G_{\xi,\lambda}(x, y)] = \sum_{j \in \mathbb{Z}} \mathcal{G}_{\xi,\lambda}(x, y + j),
 \tag{3.23}$$

which is readily seen to converge (by exponential decay in $|x - y|$) for λ in the resolvent set, and clearly is periodic and satisfies the resolvent equation on $[0, 1]$. Likewise, the periodic Green function G_ξ may be expressed in terms of the whole-line version \mathcal{G}_ξ , as

$$(3.24) \quad [G_\xi(x, t; y)] = \sum_{j \in \mathbb{Z}} \mathcal{G}_\xi(x, t; y + j).$$

See (5.14)–(5.15) for an illustrative computation in the scalar constant-coefficient case. This clarifies the results obtained above by a direct computation, and the relation between the periodic and whole-line kernels. Here, by the “whole-line” version, we mean the kernel of periodic-coefficient operator considered as acting on $L^2(\mathbb{R})$.

4 Pointwise bounds on G

Now we start the pointwise bounds on G . Let's first define the sector

$$\Omega := \{\lambda : \operatorname{Re}(\lambda) \leq \theta_1 - \theta_2 |\operatorname{Im}(\lambda)|\},$$

where θ_1 and $\theta_2 > 0$ are small constants.

Proposition 4.1 ([ZH]). *The parabolic operator $\partial_t - L$ has a Green function $G(x, t; y)$ for each fixed y and $(x, t) \neq (y, 0)$ given by*

$$(4.1) \quad G(x, t; y) = \frac{1}{2\pi i} \int_{\Gamma := \partial(\Omega \setminus B(0, R))} e^{\lambda t} G_\lambda(x, y) d\lambda$$

for $R > 0$ sufficiently large and $\theta_1, \theta_2 > 0$ sufficiently small. This is the standard spectral resolution (inverse Laplace transform) formula.

Proof of Theorem 1.2. Case(i). $\frac{|x - y|}{t}$ **large.** We first consider the case that $|x - y|/t \geq S$, S sufficiently large. For this case, it is hard to estimate G through $|[G_\xi(x, t; y)]|$, directly, because of the problem of aliasing; see Remark 4.2. Instead we estimate $|G_\lambda(x, y)|$ first and we estimate $|G(x, t; y)|$ by (4.1). This is treated by exactly the same argument as in [ZH]. By [ZH], notice that

$$|G_\lambda(x, y)| \leq C |\lambda|^{-1/2} |e^{-\beta^{-1/2} |\lambda|^{1/2} |x - y|}|,$$

for all $\lambda \in \Omega \setminus B(0, R)$ and $R > 0$ sufficiently large, and here, $\beta^{-1/2} \sim \min_{\lambda \in \Omega \cap \{|\lambda| > R\}} \operatorname{Re} \sqrt{\lambda/|\lambda|}$.

Finally we have

$$|G(x, t; y)| \leq C \left| \int_{\Gamma} e^{\lambda t} G_\lambda(x, y) d\lambda \right| \leq t^{-\frac{1}{2}} e^{-\eta t} e^{-\frac{|x - y - at|^2}{Mt}},$$

for some $\eta > 0$ and $M > 0$ sufficiently large. (See [ZH] for a detail proof)

Case (ii). $\frac{|x-y|}{t} < S$ **bounded.** To begin, notice that by standard spectral perturbation theory [K], the total eigenprojection $P(\xi)$ onto the eigenspace of L_ξ associated with the eigenvalues $\lambda(\xi)$ bifurcating from the $(\xi, \lambda(\xi)) = (0, 0)$ state is well defined and analytic in ξ for ξ sufficiently small, since the discreteness of the spectrum of L_ξ implies that the eigenvalue $\lambda(\xi)$ is separated at $\xi = 0$ from the remainder of the spectrum of L_0 . By (D2), there exists an $\varepsilon > 0$ such that $\operatorname{Re}\sigma(L_\xi) \leq -\theta|\xi|^2$ for $0 < |\xi| < 2\varepsilon$. With this choice of ε , we first introduce a smooth cut off function $\phi(\xi)$ such that

$$\phi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \varepsilon \\ 0, & \text{if } |\xi| \geq 2\varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a sufficiently small parameter. Now from the inverse Bloch-Fourier transform representation, we split the Green function

$$G(x, t; y) = \int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t} \hat{\delta}_y(\xi, x) d\xi$$

into its low-frequency part

$$I = \int_{-\pi}^{\pi} e^{i\xi x} \phi(\xi) P(\xi) e^{L_\xi t} \hat{\delta}_y(\xi, x) d\xi$$

and high frequency part

$$II = \int_{-\pi}^{\pi} e^{i\xi x} (1 - \phi(\xi) P(\xi)) e^{L_\xi t} \hat{\delta}_y(\xi, x) d\xi.$$

Let's start by considering the first part I .

(4.2)

$$\begin{aligned} I &= \int_{|\xi| \leq 2\varepsilon} e^{i\xi x} \phi(\xi) P(\xi) e^{L_\xi t} \hat{\delta}_y(\xi, x) d\xi \\ &= \int_{|\xi| \leq 2\varepsilon} e^{i\xi x} \phi(\xi) e^{\lambda(\xi)t} q(x, \xi) \tilde{q}(y, \xi) d\xi \\ &= \int_{-\infty}^{\infty} e^{i\xi(x-y)} e^{(-ia\xi - b\xi^2)t} q(x, 0) \tilde{q}(y, 0) d\xi - \int_{|\xi| \geq 2\varepsilon} e^{i\xi(x-y)} e^{(-ia\xi - b\xi^2)t} q(x, 0) \tilde{q}(y, 0) d\xi \\ &\quad + \int_{|\xi| \leq 2\varepsilon} e^{i\xi(x-y)} e^{(-ia\xi - b\xi^2)t} (e^{O(|\xi|^3)t} \phi(\xi) q(x, \xi) \tilde{q}(y, \xi) - q(x, 0) \tilde{q}(y, 0)) d\xi \\ &= \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} q(x, 0) \tilde{q}(y, 0) + II' + III'. \end{aligned}$$

View II' and III' as complex contour integrals in the variable ξ and define

$$(4.3) \quad \bar{\alpha} := \left| \frac{x-y-at}{2bt} \right|$$

which is bounded because $|x - y|/t$ is bounded. Using the Cauchy's Theorem and writing $\xi_1 = \xi + i\bar{\alpha}$ and $\xi_2 = \varepsilon + iz$, we have the estimate

$$\begin{aligned}
|II'| &\leq C \left| \int_{\varepsilon}^{\infty} e^{i\xi_1(x-y-at)} e^{-b\xi_1^2 t} d\xi_1 \right| + C \left| \int_0^{\bar{\alpha}} e^{i\xi_2(x-y-at)} e^{-b\xi_2^2 t} d\xi_2 \right| \\
&= C \int_{\varepsilon}^{\infty} \left| e^{i(\xi+i\bar{\alpha})2bt\bar{\alpha}} e^{-b(\xi+i\bar{\alpha})^2 t} \right| d\xi + C \int_0^{\bar{\alpha}} \left| e^{i(\varepsilon+iz)2bt\bar{\alpha}} e^{-b(\varepsilon+iz)^2 t} \right| dz \\
&= C e^{-bt\bar{\alpha}^2} \int_{\varepsilon}^{\infty} e^{-b\xi^2 t} d\xi + C e^{-b\varepsilon^2 t} \int_0^{\bar{\alpha}} e^{btz^2 - 2bt\bar{\alpha}z} dz \\
&\leq C e^{-\frac{|x-y-at|^2}{4bt}} t^{-\frac{1}{2}} e^{-\eta t} + C e^{-b\varepsilon^2 t} \int_0^{\bar{\alpha}} e^{-btz^2} dz \\
&\leq C e^{-\frac{|x-y-at|^2}{4bt}} t^{-\frac{1}{2}} e^{-\eta t} + C e^{-b\varepsilon^2 t} t^{-\frac{1}{2}} e^{-\eta t} \\
&\leq C t^{-\frac{1}{2}} e^{-\eta t} e^{-\frac{|x-y-at|^2}{Mt}},
\end{aligned}$$

for some positive η and $M > 0$ sufficiently large.

Similarly, setting

$$\tilde{\alpha} = \min\{\varepsilon, \bar{\alpha}\},$$

we can estimate $|III'|$ which is

$$\begin{aligned}
&|III'| \\
&= C \left| \int_{|\xi| \leq \varepsilon} e^{i\xi(x-y)} e^{(-ia\xi - b\xi^2)t} \left(e^{O(|\xi|^3)t} q(x, \xi) \tilde{q}(y, \xi) - q(x, 0) \tilde{q}(y, 0) \right) d\xi \right| \\
&\leq C \left| \int_{|\xi| \leq \varepsilon} e^{i\xi(x-y)} e^{(-ia\xi - b\xi^2)t} \left(e^{O(|\xi|^3)t} - 1 + O(|\xi|) \right) d\xi \right| \\
&\leq C \int_{-\varepsilon}^{\varepsilon} \left| e^{i(\xi+i\bar{\alpha})(x-y-at)} e^{-b(\xi+i\bar{\alpha})^2 t} \left(e^{O(|\xi|^3)t + O(|\bar{\alpha}|^3)t} - 1 + O(|\xi|) + O(|\bar{\alpha}|) \right) \right| d\xi \\
&\quad + C \int_0^{\bar{\alpha}} \left| e^{i(\varepsilon+iz)(x-y-at)} e^{-b(\varepsilon+iz)^2 t} \left(e^{O(|\varepsilon|^3)t + O(|z|^3)t} - 1 + O(|\varepsilon|) + O(|z|) \right) \right| dz \\
&\leq C e^{-bt\bar{\alpha}^2} \int_{-\varepsilon}^{\varepsilon} e^{-b\xi^2 t} \left(e^{O(|\xi|^3)t + O(|\bar{\alpha}|^3)t} + O(|\xi|) + 1 \right) d\xi \\
&\quad + C e^{-b\varepsilon^2 t} \int_0^{\bar{\alpha}} e^{bz^2 t - 2bt\bar{\alpha}z} \left(e^{O(|\varepsilon|^3)t + O(|z|^3)t} + O(|z|) + 1 \right) dz \\
&\leq C e^{-\frac{bt\bar{\alpha}^2}{2}} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{b\xi^2 t}{2}} \left(O(|\xi|) + 1 \right) d\xi + C e^{-\frac{b\varepsilon^2 t}{2}} \int_0^{\bar{\alpha}} e^{-\frac{bz^2 t}{2}} \left(O(|z|) + 1 \right) dz \\
&\leq C e^{-\frac{bt\bar{\alpha}^2}{2}} \left(\int_{-\varepsilon}^{\varepsilon} e^{-\frac{b\xi^2 t}{2}} |\xi| d\xi + \int_{-\varepsilon}^{\varepsilon} e^{-\frac{b\xi^2 t}{2}} d\xi \right) + C e^{-\frac{b\varepsilon^2 t}{2}} \left(\int_0^{\bar{\alpha}} e^{-\frac{bz^2 t}{2}} |z| dz + \int_0^{\bar{\alpha}} e^{-\frac{bz^2 t}{2}} dz \right) \\
&\leq C e^{-\frac{|x-y-at|^2}{M_2 t}} \left((t+1)^{-1} + t^{-\frac{1}{2}} e^{-\eta t} \right),
\end{aligned}$$

for some $\eta > 0$ and $M > 0$ sufficiently large.

Next, we consider the second part *II*. Noting first that

$$\hat{\delta}_y(\xi, x) = \sum_{j \in \mathbb{Z}} e^{j2\pi i x} \hat{\delta}_y(\xi + j2\pi) = \sum_{j \in \mathbb{Z}} e^{j2\pi i x} e^{-(\xi + j2\pi)y} = e^{-i\xi y} \sum_{j \in \mathbb{Z}} e^{j2\pi i(x-y)} = e^{-i\xi y} [\delta_y(x)],$$

we have for $|\xi| \geq 2\varepsilon$, $\phi(\xi) = 0$ and

$$\begin{aligned} & \int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi x} (1 - \phi(\xi)P(\xi)) e^{L_\xi t} \hat{\delta}_y(\xi, x) d\xi \\ &= \int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi x} e^{L_\xi t} \hat{\delta}_y(\xi, x) d\xi \\ &= \int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi(x-y)} e^{L_\xi t} [\delta_y(x)] d\xi \\ &= \int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi(x-y)} [G_\xi(x, t; y)] d\xi, \end{aligned}$$

where the brackets $[\cdot]$ denote the periodic extensions of the given function onto the whole line. Assuming that $\operatorname{Re} \sigma(L_\xi) \leq -\eta < 0$ for $|\xi| \geq 2\varepsilon$, we have

$$[G_\xi(x, t; y)] = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t} [G_{\xi, \lambda}(x, y)] d\lambda,$$

here, we fix $\Gamma_1 = \partial(\Omega \cap \{Re \lambda \leq -\eta\})$ independent of ξ . Parameterizing Γ_1 by $Im \lambda := k$, and applying the bounds of $\sup_{|\xi| \leq \pi} |[G_{\xi, \lambda}(x, y)]| < O(|\lambda|^{-\frac{1}{2}})$ for large $|\lambda|$ in Section 3, we have

$$\begin{aligned} |[G_\xi(x, t; y)]| &\leq C \int_{\Gamma_1} e^{Re \lambda t} |[G_{\xi, \lambda}(x, y)]| d\lambda \\ &\leq C e^{-\eta t} \int_0^\infty k^{-\frac{1}{2}} e^{-\theta_2 k t} dk \\ &\leq C t^{-\frac{1}{2}} e^{-\eta t} \\ &\leq C t^{-\frac{1}{2}} e^{-\frac{\eta}{2} t} e^{-\frac{|x-y-at|^2}{Mt}}, \end{aligned}$$

here, the last inequality is from $\frac{|x-y-at|}{t} \leq S_1$ bounded. Indeed, for large $M > 0$,

$$e^{-\frac{|x-y-at|^2}{Mt}} = e^{-(\frac{|x-y-at|}{t})^2 \frac{t}{M}} \geq e^{-\frac{S_1}{M} t} \geq e^{-\frac{\eta}{2} t},$$

and so,

$$\begin{aligned} & \left| \int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi x} (1 - \phi(\xi)P(\xi)) e^{L_\xi t} \hat{\delta}_y(\xi, x) d\xi \right| \\ (4.4) \quad & \leq C \sup_{2\varepsilon \leq |\xi| \leq \pi} |[G_\xi(x, t; y)]| \\ & \leq C t^{-\frac{1}{2}} e^{-\frac{\eta}{2} t} e^{-\frac{|x-y-at|^2}{Mt}}. \end{aligned}$$

For $|\xi|$ sufficiently small, on the other hand, $\phi(\xi) = 1$, and $I - \phi(\xi)P = I - P = Q$, where Q is the eigenprojection of L_ξ associated with eigenvalues complementary to $\lambda(\xi)$ bifurcating from $(\xi, \lambda(\xi)) = (0, 0)$, which have real parts strictly less than zero. So we can estimate for $|\xi| \leq \varepsilon$ in the same way as in (4.4). Combining these observations, we have the estimate

$$|II| \leq Ct^{-\frac{1}{2}} e^{-\frac{\eta}{2}t} e^{-\frac{|x-y-at|^2}{Mt}},$$

for some $\eta > 0$ and sufficiently large $M > 0$.

This completes the proof of the theorem. \square

Remark 4.2. From (3.24), we see that estimating G using $[[G_\xi]]$ would result rather in the sum of aliased versions of the Green functions on the whole line, centered at all $y + j$, which for small $|x - y|/t$ would lead to non-negligible errors. That is, in the “small-time” regime $|x - y|/t$ large there is considerable cancellation in the inverse Bloch transform involving the integration with respect to ξ , that cannot be detected by modulus bounds alone. It is for this reason that we compute in this regime using direct inverse Laplace transform estimates as in [ZH]. That is, this part of our analysis has a very different flavor from the rest of the estimates using Bloch decomposition. For short time, these estimates may be obtained from standard parametrix estimates as in [F]; indeed, we conjecture that with further effort one might recover by parametrix methods the same bounds for all $|x - y|/t$ sufficiently large.

5 Example (constant-coefficient scalar case)

In this section, we illustrate the previous analysis by a simple example. Consider the constant-coefficient scalar case

$$(5.1) \quad u_t + au_x = u_{xx}, \quad a > 0 \quad \text{constant}$$

This gives a eigenvalue equation for each $\xi \in [-\pi, \pi)$,

$$(5.2) \quad u'' - (a - i2\xi)u' - (\xi^2 + ia\xi)u = \lambda u$$

Rewriting as a first-order system

$$(5.3) \quad U' = \mathbb{A}_\xi(x, \lambda)U,$$

where

$$(5.4) \quad U = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad \mathbb{A}_\xi = \begin{pmatrix} 0 & 1 \\ \lambda + \xi^2 + ia\xi & a - i2\xi \end{pmatrix}.$$

By a direct calculation we can find two eigenvalues of \mathbb{A}_ξ ,

$$(5.5) \quad \mu_\pm = \frac{a - i2\xi \pm \sqrt{a^2 + 4\lambda}}{2},$$

which are solutions of the characteristic equation

$$(5.6) \quad \mu^2 - (a - i2\xi)\mu - \lambda - \xi^2 - ia\xi = 0.$$

Without loss generality we assume $\operatorname{Re}\mu_- < 0$ and $\operatorname{Re}\mu_+ > 0$.

Let's construct $G_{\xi,\lambda}(x, y)$ and $\mathcal{G}_{\xi,\lambda}(x, y)$. To find $\mathcal{G}_{\xi,\lambda}(x, y)$, set

$$(5.7) \quad \mathcal{G}_{\xi,\lambda}(x, y) = \begin{cases} A(y)e^{\mu_-x}, & x > y, \\ B(y)e^{\mu_+x}, & x \leq y, \end{cases}$$

which satisfies the jump condition $\left[\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix} \right] \Big|_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By a direct calculation, we have

$$(5.8) \quad \mathcal{G}_{\xi,\lambda}(x, y) = \begin{cases} \frac{e^{\mu_-(x-y)}}{\mu_- - \mu_+}, & x > y, \\ \frac{e^{\mu_+(x-y)}}{\mu_- - \mu_+}, & x \leq y, \end{cases}$$

In this case, the projections are

$$(5.9) \quad \Pi_{\xi}^+ = \begin{pmatrix} -\frac{\mu_+}{\mu_- - \mu_+} & \frac{1}{\mu_- - \mu_+} \\ -\frac{\mu_- \mu_+}{\mu_- - \mu_+} & \frac{\mu_-}{\mu_- - \mu_+} \end{pmatrix}, \quad \Pi_{\xi}^- = \begin{pmatrix} \frac{\mu_-}{\mu_- - \mu_+} & -\frac{1}{\mu_- - \mu_+} \\ \frac{\mu_- \mu_+}{\mu_- - \mu_+} & -\frac{\mu_-}{\mu_- - \mu_+} \end{pmatrix},$$

and the solution operator of (5.3) is

$$(5.10) \quad \mathcal{F}_{\xi}^{y \rightarrow x} = e^{\mathbb{A}_{\xi}(x-y)} = e^{\mu_-(x-y)}\Pi_{\xi}^+ + e^{\mu_+(x-y)}\Pi_{\xi}^-,$$

and hence the formula (2.5) is exactly the the same as (5.8).

Similarly, we find $G_{\xi,\lambda}(x, y)$ by setting

$$(5.11) \quad G_{\xi,\lambda}(x, y) = \begin{cases} A(y)e^{\mu_-x} + B(y)e^{\mu_+x}, & x > y, \\ C(y)e^{\mu_-x} + D(y)e^{\mu_+x}, & x \leq y. \end{cases}$$

We need to find $A(y)$, $B(y)$, $C(y)$ and $D(y)$ which satisfy the periodicity $\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix}(0, y) = \begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix}(1, y)$ and the jump condition $\left[\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix} \right] \Big|_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By a direct calculation, we find for each $\xi \in [-\pi, \pi)$,

$$(5.12) \quad G_{\xi,\lambda}(x, y) = \begin{cases} \frac{e^{\mu_-(x-y)}}{(\mu_- - \mu_+)(1 - e^{\mu_-})} - \frac{e^{\mu_+(x-y)}}{(\mu_- - \mu_+)(1 - e^{\mu_+})}, & x > y, \\ \frac{e^{\mu_-(x-y+1)}}{(\mu_- - \mu_+)(1 - e^{\mu_-})} - \frac{e^{\mu_+(x-y+1)}}{(\mu_- - \mu_+)(1 - e^{\mu_+})}, & x \leq y. \end{cases}$$

To verify (2.6), we first check

$$(I - e^{\mathbb{A}_\xi})\left(\frac{1}{1 - e^{\mu_-}}\Pi_\xi^+ + \frac{1}{1 - e^{\mu_+}}\Pi_\xi^-\right) = I.$$

So

$$M_\xi^+ = (I - \mathcal{F}_\xi^{y \rightarrow y+1})^{-1} = (I - e^{\mathbb{A}_\xi})^{-1} = \frac{1}{1 - e^{\mu_-}}\Pi_\xi^+ + \frac{1}{1 - e^{\mu_+}}\Pi_\xi^-,$$

and

$$M_\xi^- = -(I - \mathcal{F}_\xi^{y \rightarrow y+1})^{-1}\mathcal{F}_\xi^{y \rightarrow y+1} = -\frac{e^{\mu_-}}{1 - e^{\mu_-}}\Pi_\xi^+ - \frac{e^{\mu_+}}{1 - e^{\mu_+}}\Pi_\xi^-.$$

This implies (2.6) is exactly the same as (5.12).

Now let's show that

$$(5.13) \quad G_{\xi,\lambda}(x, y) = \sum_{j \in \mathbb{Z}} \mathcal{G}_{\xi,\lambda}(x, y + j).$$

We first consider the case of $0 \leq y \leq x \leq 1$. For $j \leq 0$, $x > y + j$, and for $j \geq 1$, $x < y + j$. Thus we have, by the geometric series,

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \mathcal{G}_{\xi,\lambda}(x, y + j) &= \sum_{j \leq 0} \mathcal{G}_{\xi,\lambda}(x, y + j) + \sum_{j \geq 1} \mathcal{G}_{\xi,\lambda}(x, y + j) \\ &= \frac{1}{\mu_- - \mu_+} \sum_{j \leq 0} e^{\mu_-(x-y-j)} + \frac{1}{\mu_- - \mu_+} \sum_{j \geq 1} e^{\mu_+(x-y-j)} \\ &= \frac{e^{\mu_-(x-y)}}{\mu_- - \mu_+} \sum_{j \geq 0} (e^{\mu_-})^j + \frac{e^{\mu_+(x-y)}}{\mu_- - \mu_+} \sum_{j \geq 1} (e^{-\mu_+})^j \\ (5.14) \quad &= \frac{e^{\mu_-(x-y)}}{(\mu_- - \mu_+)(1 - e^{\mu_-})} + \frac{e^{\mu_+(x-y-1)}}{(\mu_- - \mu_+)(1 - e^{-\mu_+})} \\ &= \frac{e^{\mu_-(x-y)}}{(\mu_- - \mu_+)(1 - e^{\mu_-})} - \frac{e^{\mu_+(x-y)}}{(\mu_- - \mu_+)(1 - e^{\mu_+})} \\ &= G_{\xi,\lambda}(x, y). \end{aligned}$$

Similarly, we consider the case of $0 \leq x \leq y \leq 1$. For $j \leq -1$, $x > y + j$, and for $j \geq 0$,

$x \leq y + j$.

$$\begin{aligned}
 \sum_{j \in \mathbb{Z}} \mathcal{G}_{\xi, \lambda}(x, y + j) &= \sum_{j \leq -1} \mathcal{G}_{\xi, \lambda}(x, y + j) + \sum_{j \geq 0} \mathcal{G}_{\xi, \lambda}(x, y + j) \\
 &= \frac{1}{\mu_- - \mu_+} \sum_{j \leq -1} e^{\mu_-(x-y-j)} + \frac{1}{\mu_- - \mu_+} \sum_{j \geq 0} e^{\mu_+(x-y-j)} \\
 &= \frac{e^{\mu_-(x-y)}}{\mu_- - \mu_+} \sum_{j \geq 1} (e^{\mu_-})^j + \frac{e^{\mu_+(x-y)}}{\mu_- - \mu_+} \sum_{j \geq 0} (e^{-\mu_+})^j \\
 &= \frac{e^{\mu_-(x-y+1)}}{(\mu_- - \mu_+)(1 - e^{\mu_-})} + \frac{e^{\mu_+(x-y)}}{(\mu_- - \mu_+)(1 - e^{-\mu_+})} \\
 &= \frac{e^{\mu_-(x-y+1)}}{(\mu_- - \mu_+)(1 - e^{\mu_-})} - \frac{e^{\mu_+(x-y+1)}}{(\mu_- - \mu_+)(1 - e^{\mu_+})} \\
 &= G_{\xi, \lambda}(x, y).
 \end{aligned} \tag{5.15}$$

Thus, $[G_{\xi, \lambda}(x, y)] = \sum_{j \in \mathbb{Z}} \mathcal{G}_{\xi, \lambda}(x, y + j)$, and so $[G_{\xi}(x, t; y)] = \sum_{j \in \mathbb{Z}} \mathcal{G}_{\xi}(x, t; y + j)$ for all $x, y \in \mathbb{R}$.

6 Behavior of u for $u_t = u_{xx} + u^q$, $q \geq 4$

In this section, we start with the nonlinear analysis of a perturbed heat equation as practice for our later analysis of $u_t = Lu + O(|u|^2)$ for the linear operator L of (1.2). We show the behavior of u satisfying $u_t = u_{xx} + u^q$, $q \geq 4$ for three cases of initial data ($u_0(x) = u(x, 0)$):

- (1) $|u_0|_{L^1 \cap L^\infty}, |xu_0|_{L^1} \leq E_0$,
- (2) $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{Mt}}$,
- (3) $|u_0(x)| \leq E_0(1 + |x|)^{-r}$, $r > 2$,

where $E_0 > 0$ is sufficiently small and $M > 0$ sufficiently large. It is very natural to consider only $q \geq 4$ because for heat kernel k , $u^q \sim k^q \sim t^{-\frac{(q-1)}{2}}k$ and $u_t, u_{xx} \sim t^{-1}k$ implies that $\frac{(q-1)}{2} > 1$ is the criterion that the nonlinear part be asymptotically negligible; see [S1, S2] for further discussion.

6.1 Behavior for initial data $|u_0|_{L^1 \cap L^\infty}, |xu_0|_{L^1} \leq E_0$

In this section, we take $E_0 > 0$ sufficiently small and $q \geq 4$.

Lemma 6.1. *Suppose that $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0|_{L^1 \cap L^\infty} \leq E_0$, for $E_0 > 0$ sufficiently small and $q \geq 4$. Define*

$$\zeta(t) := \sup_{0 \leq s \leq t, 1 \leq p \leq \infty} |u|_{L^p}(s)(1+s)^{\frac{1}{2}(1-\frac{1}{p})}.$$

Then, for all $t \geq 0$ for which $\zeta(t)$ is finite, some $C > 0$,

$$(6.1) \quad \zeta(t) \leq C(E_0 + \zeta^4(t)).$$

Proof. Noting, because of $q \geq 4$, that

$$|u|_{L^\infty}(s) \leq \zeta(t)(1+s)^{-\frac{1}{2}} \quad \text{and} \quad |u^q|_{L^1(x)}(s) \leq |u^{q-1}|_{L^\infty}|u|_{L^1} \leq \zeta^4(t)(1+s)^{-\frac{3}{2}},$$

we obtain

$$\begin{aligned} |u(\cdot, t)|_{L^p(x)} &\leq \left| \int_{-\infty}^{\infty} k(x-y, t) u_0(y) dy \right|_{L^p(x)} + \left| \int_0^t \int_{-\infty}^{\infty} k(x-y, t-s) u^q(y, s) dy ds \right|_{L^p(x)} \\ &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} + C\zeta^4(t) \int_0^t (1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})} (1+s)^{-\frac{3}{2}} ds \\ &\leq C(E_0 + \zeta^4(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}. \end{aligned}$$

Rearranging, we obtain (6.1). \square

Corollary 6.2. *Suppose that $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0|_{L^1 \cap L^\infty} \leq E_0$, for $E_0 > 0$ sufficiently small and $q \geq 4$. Then*

$$(6.2) \quad |u(x, t)|_{L^p(x)} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}.$$

Proof. Recalling that $\zeta(t)$ is continuous so long as it remains finite, it follows by continuous induction that $\zeta(t) \leq 2CE_0$ for all $t \geq 0$ provided $E_0 < \left(\frac{1}{2c}\right)^{\frac{4}{3}}$ and (as holds without loss of generality) $C \geq 1$, and hence (6.1) implies (6.2). \square

Lemma 6.3. *Let $u(x, t)$ satisfy $u_t = u_{xx} + u^q$ and $|u_0|_{L^1 \cap L^\infty}, |xu_0|_{L^1} \leq E_0$. Define*

$$\zeta(t) := \sup_{0 \leq s \leq t} |xu(x, s)|_{L^1(x)} (1+s)^{-\frac{1}{2}}.$$

Then, for all $t \geq 0$ for which $\zeta(t)$ is finite, some $C > 0$,

$$(6.3) \quad \zeta(t) \leq C(E_0 + \zeta^2(t)).$$

Proof. Noting, by (6.2) and $q \geq 4$, that

$$|xu^q(x, t)|_{L^1(x)} \leq |u^{q-1}(x, t)|_{L^\infty} |xu(x, t)|_{L^1} \leq CE_0 \zeta(t) (1+t)^{-\frac{q-1}{2} + \frac{1}{2}} \leq CE_0 \zeta(t) (1+t)^{-1},$$

we obtain the estimate

$$\begin{aligned}
& |xu(x, t)|_{L^1(x)} \\
& \leq \left| \int_{-\infty}^{\infty} \frac{x}{\sqrt{t}} e^{-\frac{|x-y|^2}{t}} u_0(y) dy \right|_{L^1(x)} + \left| \int_0^t \int_{-\infty}^{\infty} \frac{x}{\sqrt{t-s}} e^{-\frac{|x-y|^2}{t-s}} u^q(y, s) dy ds \right|_{L^1(x)} \\
& \leq \left| \int_{-\infty}^{\infty} \left(\frac{x-y}{\sqrt{t}} e^{-\frac{|x-y|^2}{t}} u_0(y) + \frac{y}{\sqrt{t}} e^{-\frac{|x-y|^2}{t}} u_0(y) \right) dy \right|_{L^1(x)} \\
& \quad + \left| \int_0^t \int_{-\infty}^{\infty} \frac{x-y}{\sqrt{t-s}} e^{-\frac{|x-y|^2}{t-s}} u^q(y, s) + \frac{y}{\sqrt{t-s}} e^{-\frac{|x-y|^2}{t-s}} u^q(y, s) dy ds \right|_{L^1(x)} \\
& \leq C \left((1+t)^{\frac{1}{2}} |u_0|_{L^1} + |xu_0|_{L^1} \right) + C \int_0^t \left((1+t-s)^{\frac{1}{2}} |u^q(x, s)|_{L^1} + |xu^q(x, s)|_{L^1} \right) ds \\
& \leq CE_0(1+t)^{\frac{1}{2}} + CE_0 \int_0^t (1+t-s)^{\frac{1}{2}} (1+s)^{-\frac{3}{2}} ds + CE_0 \zeta(t) \int_0^t (1+s)^{-1} ds \\
& \leq C(E_0 + \zeta^2(t))(1+t)^{\frac{1}{2}}.
\end{aligned}$$

Rearranging, we obtain (6.3). □

Corollary 6.4. *Let $u(x, t)$ satisfy $u_t = u_{xx} + u^q$ and $|u_0|_{L^1 \cap L^\infty}$, $|xu_0|_{L^1} \leq E_0$, for $E_0 > 0$ sufficiently small, and $q \geq 4$. Then*

$$(6.4) \quad |xu(x, t)|_{L^1} \leq CE_0(1+t)^{\frac{1}{2}}, \quad \text{for all } t \geq 0.$$

Proof. Recalling that $\zeta(t)$ is continuous so long as it remains finite, it follows by continuous induction that $\zeta(t) \leq 2CE_0$ for all $t \geq 0$ provided $E_0 < \frac{1}{4C^2}$ and (as holds without loss of generality) $C \geq 1$, and hence (6.3) implies (6.4). □

Lemma 6.5. *Suppose that $u(x, t)$ solves $u_t = u_{xx}$ and $|u_0|_{L^1 \cap L^\infty}$, $|xu_0|_{L^1} \leq E_0$. Then*

$$(6.5) \quad |u(x, t) - U_0 k(x, t)|_{L^p(x)} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}},$$

where $U_0 := \int_{-\infty}^{\infty} u_0(x) dx$ and $k(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$.

Proof. Setting $e(x, t) := u(x, t) - U_0 k(x, t)$, we have

$$e_t(x, t) = e_{xx}(x, t) \quad \text{and} \quad \int_{-\infty}^{\infty} e_0(x) dx = 0,$$

so that, for any $t \geq 0$,

$$\begin{aligned}
(6.6) \quad |e(x, t)|_{L^p(x)} &= \left| \int_{-\infty}^{\infty} k(x-y, t) e_0(y) dy \right|_{L^p(x)} \\
&\leq \int_{-\infty}^{\infty} |k(x-y, t)|_{L^p(x)} |u_0(y)| dy + |U_0| |k(x, t)|_{L^p(x)} \\
&\leq 2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} |u_0|_{L^1}.
\end{aligned}$$

For $t \leq 1$, $\sqrt{2}(1+t)^{-\frac{1}{2}} > 1$, and hence, (6.6) implies

$$|u(x, t) - U_0 k(x, t)|_{L^p(x)} \leq 2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} |u_0|_{L^1} \leq 2\sqrt{2}E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$

Now we consider the case of $t > 1$. Noting, by the Mean Value Theorem, that

$$|k(x-y, t) - k(x, t)|_{L^p(x)} \leq |y| \int_0^1 |k_x(x-wy, t)|_{L^p(x)} dw \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} |y|,$$

we obtain

$$\begin{aligned} |u(x, t) - U_0 k(x, t)|_{L^p(x)} &\leq \int_{-\infty}^{\infty} |k(x-y) - k(x, t)|_{L^p(x)} |u_0(y)| dy \\ &\leq Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \int_{-\infty}^{\infty} |y| |u_0(y)| dy \\ &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \end{aligned}$$

□

Lemma 6.6. Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0|_{L^1 \cap L^\infty}, |xu_0|_{L^1} \leq E_0$, for $E_0 > 0$ sufficiently small and $q \geq 4$. Then

$$(6.7) \quad \left| \int_{-\infty}^{\infty} k(x-y, t-s) u^q(y, s) dy - U(s) k(x, t-s) \right|_{L^p(x)} \leq CE_0(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} (1+s)^{-1},$$

where $U(s) = \int_{-\infty}^{\infty} u^q(y, s) dy$.

Proof. Noting first, by (6.2) and (6.4), that

$$|xu^q(x, t)|_{L^1(x)} \leq |u^{q-1}|_{L^\infty(x)} |xu(x, t)|_{L^1(x)} \leq CE_0(1+t)^{-\frac{3}{2}} (1+t)^{\frac{1}{2}} \leq CE_0(1+t)^{-1},$$

we have

$$\begin{aligned} (6.8) \quad &\left| \int_{-\infty}^{\infty} k(x-y, t-s) u^q(y, s) dy - U(s) k(x, t-s) \right|_{L^p(x)} \\ &\leq \int_{-\infty}^{\infty} |k(x-y, t-s) - k(x, t-s)|_{L^p(x)} |u^q(y, s)| dy \\ &\leq |k_x(x-y^*, t-s)|_{L^p(x)} |yu^q(y, s)|_{L^1(y)} \\ &\leq CE_0(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} (1+s)^{-1}. \end{aligned}$$

□

Theorem 6.7 (Behavior). *Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0|_{L^1 \cap L^\infty}, |xu_0|_{L^1} \leq E_0$, for $E_0 > 0$ sufficiently small and $q \geq 4$. Set*

$$U_* = \int_0^\infty U(s)ds + U_0 = \int_0^\infty \int_{-\infty}^\infty u^q(y, s)dyds + \int_{-\infty}^\infty u_0(y)dy.$$

Then $|U_*| < \infty$ and

$$(6.9) \quad |u(x, t) - U_*k(x, t)|_{L^p(x)} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+\ln(1+t)).$$

Proof. Noting first, by (6.2) and $q \geq 4$, that

$$(6.10) \quad |U(s)| = |u^q|_{L^1} = |u|_{L^q}^q \leq CE_0(1+s)^{-\frac{3}{2}},$$

we obtain

$$|U_*| \leq CE_0 \int_0^\infty (1+s)^{-\frac{3}{2}} + |u_0|_{L^1} < \infty.$$

Now we break $|u(x, t) - U_*k(x, t)|_{L^p(x)}$ into four parts.

$$\begin{aligned} |u(x, t) - U_*k(x, t)|_{L^p(x)} &\leq \left| \int_{-\infty}^\infty k(x-y, t)u_0(y)dy - \int_{-\infty}^\infty k(x, t)u_0(y)dy \right|_{L^p(x)} \\ &\quad + \int_t^\infty |U(s)||k(x, t)|_{L^p(x)}ds \\ &\quad + \int_0^t \left| \int_{-\infty}^\infty k(x-y, t-s)u^q(y, s)dy - k(x, t-s)U(s) \right|_{L^p(x)} ds \\ &\quad + \int_0^t |U(s)||k(x, t-s) - k(x, t)|_{L^p(x)}ds \\ &= I + II + III + IV. \end{aligned}$$

By (6.5), we already have $I \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$. By (6.10),

$$II \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \int_t^\infty (1+s)^{-\frac{3}{2}}ds \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}},$$

By (6.7), we have

$$\begin{aligned} III &\leq CE_0 \int_0^t (1+s)^{-1}(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}ds \\ &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \int_0^{t/2} (1+s)^{-1}ds + CE_0(1+t)^{-1} \int_{t/2}^t (1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}ds \\ &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+\ln(1+t)). \end{aligned}$$

By (6.10) and by the Mean Value Theorem, for some $s^* \in (0, t/2)$, we have

$$\begin{aligned}
IV &\leq CE_0 \int_{t/2}^t (1+s)^{-\frac{3}{2}} |k(x, t-s) - k(x, t)|_{L^p(x)} ds \\
&\quad + CE_0 \int_0^{t/2} (1+s)^{-\frac{3}{2}} s |k_t(x, t-s^*)|_{L^p(x)} ds \\
&\leq CE_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \int_{t/2}^t (1+s)^{-\frac{3}{2}} ds + CE_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1} \int_0^{t/2} (1+s)^{-\frac{1}{2}} ds \\
&\leq CE_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.
\end{aligned}$$

□

6.2 Behavior for initial data $|v_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$

In this section, we take $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large and $q \geq 4$. We start with the following lemma which is a very useful calculation for following sections.

Lemma 6.8. *For all $0 < s < t$,*

$$(6.11) \quad \int_{-\infty}^{\infty} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} s^{-\frac{1}{2}} e^{-\frac{|y|^2}{s}} dy \leq t^{-\frac{1}{2}} e^{-\frac{|x|^2}{t}}.$$

Proof. Noting first that

$$\frac{|x-y|^2}{t-s} + \frac{|y|^2}{s} = \frac{s(x^2 - 2xy + y^2) + (t-s)y^2}{s(t-s)} = \frac{t(y - \frac{sx}{t})^2 + sx^2 \frac{(t-s)}{t}}{s(t-s)},$$

we obtain

$$\int_{-\infty}^{\infty} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} s^{-\frac{1}{2}} e^{-\frac{|y|^2}{s}} dy \leq e^{-\frac{|x|^2}{t}} \int_{-\infty}^{\infty} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} e^{-\frac{t(y-sx/t)^2}{s(t-s)}} dy \leq t^{-\frac{1}{2}} e^{-\frac{|x|^2}{t}}.$$

□

Lemma 6.9. *Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small and $q \geq 4$. Define*

$$\zeta(t) := \sup_{0 \leq s \leq t, x \in \mathbb{R}} |u(x, s)| (1+s)^{\frac{1}{2}} e^{\frac{|x|^2}{M(1+s)}},$$

with $M > 0$ sufficiently large. Then, for all $t \geq 0$ for which $\zeta(t)$ is finite,

$$(6.12) \quad \zeta(t) \leq C(E_0 + \zeta^2(t)).$$

Proof. By $|u^q| = |u^{q-2}|_{L^\infty} |u^2| \leq \zeta^2(t)(1+s)^{-1}(1+s)^{-\frac{1}{2}} e^{-\frac{|x|^2}{M(1+s)}}$ and (6.11), we obtain

$$\begin{aligned}
& |u(x, t)| \\
& \leq \int_{-\infty}^{\infty} k(x-y, t) |u_0(y)| dy + \int_0^t \int_{-\infty}^{\infty} k(x-y, t-s) |u^q(y, s)| dy ds \\
& \leq CE_0 \int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} e^{-\frac{|y|^2}{M}} dy + C\zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} (1+s)^{-\frac{3}{2}} e^{-\frac{|x|^2}{M(1+s)}} dy ds \\
& \leq CE_0 e^{-\frac{|x|^2}{M(1+t)}} (M+t)^{-\frac{1}{2}} + C\zeta^2(t) (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{M(1+t)}} \int_0^t (1+s)^{-\frac{3}{2}} ds \\
& \leq C(E_0 + \zeta^2(t)) (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{M(1+t)}}.
\end{aligned}$$

Rearranging, we have (6.12). □

Corollary 6.10. *Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small, $M > 0$ sufficiently large, and $q \geq 4$. Then*

$$(6.13) \quad |u(x, t)| \leq CE_0 (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{M(1+t)}}$$

Proof. Same proof as Corollary 6.2. □

Lemma 6.11. *Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small, $M > 0$ sufficiently large, and $q \geq 4$. Then for $M' > M$,*

$$(6.14) \quad |xu(x, t)| \leq CE_0 e^{-\frac{|x|^2}{M'(1+t)}}.$$

Proof. Notice first that $|x|e^{-|x|^2} \leq Ce^{-|x|^2/r}$ for $r > 1$. Then by (6.13), we have

$$|xu(x, t)| \leq CE_0 |x| (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{M(1+t)}} \leq CE_0 e^{-\frac{|x|^2}{M'(1+t)}}.$$

□

Lemma 6.12. *Suppose $u(x, t)$ satisfies $u_t = u_{xx}$ and $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small, $M > 0$ sufficiently large, and $q \geq 4$. Then for some sufficiently large $M'' > M' > M$,*

$$(6.15) \quad |u(x, t) - U_0 k(x, t)| \leq CE_0 (1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}},$$

where $U_0 = \int_{-\infty}^{\infty} u_0(y) dy$ and $k(x, t) = (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}}$. (Note: $|U_0| \leq E_0 \sqrt{M}$)

Proof. Noting, by the Mean Value Theorem, that

$$|k(x-y, t) - k(x, t)| \leq |y| \int_0^1 |k_x(x-wy, t)| dw$$

we obtain

$$\begin{aligned} |u(x, t) - U_0 k(x, t)| &\leq \int_{-\infty}^{\infty} |k(x-y, t) - k(x, t)| |u_0(y)| dy \\ &\leq E_0 \int_{-\infty}^{\infty} \int_0^1 (1+t)^{-\frac{3}{2}} |x-wy| e^{-\frac{|x-wy|^2}{(1+t)}} |y| e^{-\frac{|y|^2}{M}} dw dy \\ &\leq E_0 \int_0^1 \int_{-\infty}^{\infty} (1+t)^{-1} e^{-\frac{|x-wy|^2}{M'(1+t)}} e^{-\frac{|y|^2}{M'}} dy dw \\ &\leq CE_0(1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}}. \end{aligned}$$

□

Lemma 6.13. Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small, $M > 0$ sufficiently large, and $q \geq 4$. Then for some sufficiently large $M'' > M' > M$,

$$(6.16) \quad \left| \int_{-\infty}^{\infty} k(x-y, t-s) u^q(y, s) dy - U(s) k(x, t-s) \right| \leq E_0 (1+t-s)^{-1} (1+s)^{-1} e^{-\frac{|x|^2}{M''(1+t)}},$$

where $U(s) = \int_{-\infty}^{\infty} u^q(y, s) dy$.

Proof. Noting first that by $q \geq 4$,

$$|xu^q(x, s)| \leq |u^{p-1}|_{L^\infty} |xu(x, s)| \leq CE_0(1+s)^{-\frac{3}{2}} e^{-\frac{|x|^2}{M'(1+s)}},$$

we have, by (6.11)

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} k(x-y, t-s) u^q(y, s) dy - U(s) k(x, t-s) \right| \\ &\leq \int_{-\infty}^{\infty} |k(x-y, t-s) - k(x, t-s)| |u^q(y, s)| dy \\ &\leq E_0 \int_{-\infty}^{\infty} \int_0^1 (1+t-s)^{-\frac{3}{2}} |x-wy| e^{-\frac{|x-wy|^2}{(1+t-s)}} |y| u^q(y, s) dw dy \\ &\leq E_0 \int_0^1 \int_{-\infty}^{\infty} (1+t-s)^{-1} (1+s)^{-\frac{3}{2}} e^{-\frac{|x-wy|^2}{M'(1+t-s)}} e^{-\frac{|y|^2}{M'(1+s)}} dy dw \\ &\leq CE_0(1+t-s)^{-1} (1+s)^{-1} e^{-\frac{|x|^2}{M''(1+t)}}. \end{aligned}$$

□

Theorem 6.14 (Behavior). *Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small, $M > 0$ sufficiently large, and $q \geq 4$. Set*

$$U_* = \int_0^\infty U(s) ds + U_0 = \int_0^\infty \int_{-\infty}^\infty u^q(y, s) dy ds + \int_{-\infty}^\infty u_0(y) dy.$$

Then $|U_*| < \infty$ and for some sufficiently large $M'' > M' > M$,

$$(6.17) \quad |u(x, t) - U_* k(x, t)| \leq E_0 (1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}} (1 + \ln(1+t)).$$

Proof. Recalling (6.2) and $q \geq 4$, $|U(s)| \leq CE_0(1+s)^{-\frac{3}{2}}$ and so

$$|U_*| \leq CE_0 \int_0^\infty (1+s)^{-\frac{3}{2}} ds + |u_0|_{L^1} < \infty.$$

Now we break $|u(x, t) - U_* k(x, t)|$ into four parts like (6.11). Then

$$(6.18) \quad II \leq CE_0 (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}} \int_t^\infty (1+s)^{-\frac{3}{2}} ds \leq CE_0 (1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}}.$$

By (6.16), we have

$$(6.19) \quad \begin{aligned} III &\leq CE_0 e^{-\frac{|x|^2}{2M'(1+t)}} \int_0^t (1+t-s)^{-1} (1+s)^{-1} ds \\ &\leq CE_0 (1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}} \ln(1+t). \end{aligned}$$

By $|U(s)| \leq CE_0(1+s)^{-\frac{3}{2}}$ and by the Mean Value Theorem, we have, for some $s^* \in (0, t/2)$,

$$(6.20) \quad \begin{aligned} IV &\leq \int_0^t |U(s)| |k(x, t-s) - k(x, t)| ds \\ &\leq CE_0 \int_{t/2}^t (1+s)^{-\frac{3}{2}} \left[(1+t-s)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t-s)}} + (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}} \right] ds \\ &\quad + CE_0 \int_0^{t/2} (1+s)^{-\frac{3}{2}} |s| |k_t(x, t-s^*)| ds \\ &\leq E_0 (1+t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{(1+t)}} \int_{t/2}^t (1+t-s)^{-\frac{1}{2}} ds + E_0 (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}} \int_{t/2}^t (1+s)^{-\frac{3}{2}} ds \\ &\quad + E_0 e^{-\frac{|x|^2}{(1+t)}} \int_0^{t/2} (1+s)^{-\frac{1}{2}} (1+t-s^*)^{-\frac{3}{2}} ds \\ &\leq E_0 (1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}}. \end{aligned}$$

By (6.15) and (6.18)–(6.20), we have (6.17). \square

6.3 Behavior for initial data $|v_0(x)| \leq E_0(1 + |x|)^{-r}$, $r > 2$

In this section, we take $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large and $q \geq 4$. We start with the following two lemmas.

Lemma 6.15. *For all $t \geq 0$, $x \in \mathbb{R}$ and $r > 1$,*

$$(6.21) \quad \int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1 + |y|)^{-r} dy \leq C \left[t^{-\frac{1}{2}} \wedge (1 + |y|)^{-r} + (1 + \sqrt{t})^{-1} e^{-\frac{|x|^2}{Mt}} \right].$$

Proof. We need only consider $\int_0^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1 + |y|)^{-r} dy$ by symmetry.

Notice first that

$$\int_0^{\infty} (1 + |y|)^{-r} dy \leq \frac{1}{r-1} < \infty.$$

If $x = 0$, it is trivial, from

$$\int_0^{\infty} t^{-\frac{1}{2}} e^{-\frac{|y|^2}{t}} (1 + |y|)^{-r} dy \leq C(1 + \sqrt{t})^{-1}.$$

For $x \neq 0$, we break the integration into two parts

$$\int_0^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1 + |y|)^{-r} dy = \int_0^{|x|/2} + \int_{|x|/2}^{\infty} = I + II$$

For the first integral I , if $t \leq 1$, we have

$$\int_0^{|x|/2} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1 + |y|)^{-r} dy \leq C \frac{|x|}{\sqrt{t}} e^{-\frac{|x|^2}{t}} \leq C e^{-\frac{|x|^2}{Mt}} \leq C(1 + \sqrt{t})^{-1} e^{-\frac{|x|^2}{Mt}},$$

and if $t \geq 1$, we have

$$\begin{aligned} \int_0^{|x|/2} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1 + |y|)^{-r} dy &\leq C(1 + \sqrt{t})^{-1} e^{-\frac{|x|^2}{Mt}} \int_0^{|x|/2} (1 + |y|)^{-r} dy \\ &\leq C(1 + \sqrt{t})^{-1} e^{-\frac{|x|^2}{Mt}}. \end{aligned}$$

For the second integral II , we have

$$\int_{|x|/2}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1 + |y|)^{-r} dy \leq t^{-\frac{1}{2}} \int_{|x|/2}^{\infty} (1 + |y|)^{-r} dy \leq C t^{-\frac{1}{2}},$$

or

$$\int_{|x|/2}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1 + |y|)^{-r} dy \leq (1 + |x|)^{-r} \int_{|x|/2}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} dy \leq C(1 + |x|)^{-r}.$$

□

Corollary 6.16. *For all $t \geq 0$, $x \in \mathbb{R}$ and $r > 1$,*

$$(6.22) \quad \int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1+|y|)^{-r} dy \leq C \left[(1+|x|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x|^2}{Mt}} \right],$$

Proof. By (6.21), it is enough to show that for all $x \geq 0$ and $t \geq 0$, and any $r > 1$,

$$t^{-\frac{1}{2}} \wedge (1+|x|)^{-r} \leq C \left[(1+|x|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x|^2}{Mt}} \right].$$

For $t \leq 1$, we have

$$t^{-\frac{1}{2}} \wedge (1+|x|)^{-1} = (1+|x|)^{-1} \leq C(1+|x|+1)^{-1} \leq C(1+|x|+\sqrt{t})^{-1}.$$

For $t > 1$ and $|x| \leq \sqrt{t}$, we have $e^{-\frac{|x|^2}{Mt}} \geq e^{-\frac{1}{M}} > 0$, and so

$$t^{-\frac{1}{2}} \wedge (1+|x|)^{-r} \leq C(1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{Mt}}.$$

For $t > 1$ and $|x| \geq \sqrt{t}$,

$$(1+|x|)^{-r} \leq |x|^{-r} \leq |x|^{-1} \leq t^{-\frac{1}{2}}$$

and so

$$t^{-\frac{1}{2}} \wedge (1+|x|)^{-r} = (1+|x|)^{-r} \leq C(1+|x|+|x|)^{-r} \leq C(1+|x|+\sqrt{t})^{-r}.$$

□

Lemma 6.17. *Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leq E_0(1+|x|)^{-r}$, $r > 1$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geq 4$. Define*

$$\zeta(t) := \sup_{0 \leq s \leq t, x \in \mathbb{R}} |u(x, s)| \left[(1+|x|+\sqrt{s})^{-r} + (1+\sqrt{s})^{-1} e^{-\frac{|x|^2}{M(1+s)}} \right]^{-1}.$$

Then for all $t \leq 0$ for which $\zeta(t)$ is finite, some $C > 0$,

$$(6.23) \quad \zeta(t) \leq C(E_0 + \zeta^2(t)).$$

Proof. By Duhamel's formula, we have

$$|u(x, t)| \leq \int_{-\infty}^{\infty} k(x-y, t) |u_0(y)| dy + \int_0^t \int_{-\infty}^{\infty} k(x-y, t-s) |u^q(y, s)| dy ds = I + II.$$

By (6.22), we already have $I \leq CE_0 \left[(1 + |x| + \sqrt{t})^{-r} + (1 + \sqrt{t})^{-1} e^{-\frac{|x|^2}{M(1+t)}} \right]$. Now we break II into three parts. Recalling (6.2) and $q \geq 4$, $|u^{p-2}|_{L^\infty} \leq (1+s)^{-1}$, we have

$$\begin{aligned}
II &\leq \int_0^t \int_{-\infty}^\infty (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} |u^{p-2}|_{L^\infty} |u^2(y, s)| dy ds \\
&\leq \zeta^2(t) \int_0^t \int_{-\infty}^\infty (1+s)^{-1} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} (1+|y| + \sqrt{s})^{-2r} dy ds \\
&\quad + \zeta^2(t) \int_0^t \int_{-\infty}^\infty (1+s)^{-1} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} (1+\sqrt{s})^{-2} e^{-\frac{|y|^2}{M(1+s)}} dy ds \\
&\quad + \zeta^2(t) \int_0^t \int_{-\infty}^\infty (1+s)^{-1} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} (1+|y| + \sqrt{s})^{-r} (1+\sqrt{s})^{-1} e^{-\frac{|y|^2}{M(1+s)}} dy ds \\
&= I' + II' + III'
\end{aligned}$$

Since $III' \leq CII'$, we need only estimate the two parts I' and II' . Recalling (6.11), we have

$$\begin{aligned}
II' &\leq \zeta^2(t) \int_0^t (1+s)^{-2} \int_{-\infty}^\infty (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} e^{-\frac{|y|^2}{M(1+s)}} dy ds \\
&\leq \zeta^2(t) (1 + \sqrt{t})^{-1} e^{-\frac{|x|^2}{M(1+t)}} \int_0^t (1+s)^{-2} (1+s)^{\frac{1}{2}} ds \\
&\leq \zeta^2(t) (1 + \sqrt{t})^{-1} e^{-\frac{|x|^2}{M(1+t)}}.
\end{aligned}$$

By (6.22), we break I' into two parts,

$$\begin{aligned}
I' &= \zeta^2(t) \int_0^t \int_{-\infty}^\infty (1+s)^{-1} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} (1+|y| + \sqrt{s})^{-r} (1+|y| + \sqrt{s})^{-r} dy ds \\
&\leq \zeta^2(t) \int_0^t (1+s)^{-1} (1+\sqrt{s})^{-r} \int_{-\infty}^\infty (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} (1+|y|)^{-r} dy ds \\
&\leq \zeta^2(t) \int_0^t (1+\sqrt{s})^{-r-2} \left[(1+|x| + \sqrt{t-s})^{-r} + (1+\sqrt{t-s})^{-1} e^{-\frac{|x|^2}{M(t-s)}} \right] ds \\
&= I'' + II''
\end{aligned}$$

Now we estimate I'' and II'' ,

$$\begin{aligned}
I'' &\leq C\zeta^2(t) \left[(1+|x| + \sqrt{t})^{-r} \int_0^{t/2} (1+\sqrt{s})^{-r-2} ds + (1+|x|)^{-r} \int_{t/2}^t (1+\sqrt{s})^{-r-2} ds \right] \\
&\leq C\zeta^2(t) (1+|x| + \sqrt{t})^{-r} + C\zeta^2(t) \left[(1+|x|)(1+\sqrt{t}) \right]^{-r} \\
&\leq C\zeta^2(t) (1+|x| + \sqrt{t})^{-r},
\end{aligned}$$

and

$$\begin{aligned}
II'' &\leq C\zeta^2(t)e^{-\frac{|x|^2}{Mt}} \int_0^t (1+\sqrt{s})^{-r-2}(1+\sqrt{t-s})^{-1}ds \\
&\leq C\zeta^2(t)e^{-\frac{|x|^2}{Mt}} \left[(1+\sqrt{t})^{-1} \int_0^{t/2} (1+\sqrt{s})^{-r-2}ds + (1+\sqrt{t})^{-r-2} \int_{t/2}^t (1+\sqrt{t-s})^{-1}ds \right] \\
&\leq C\zeta^2(t)(1+\sqrt{t})^{-1}e^{-\frac{|x|^2}{M(1+t)}}.
\end{aligned}$$

□

Corollary 6.18. *Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leq E_0(1+|x|)^{-r}$, $r > 1$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geq 4$. Then for all $t \geq 0$ and $x \in \mathbb{R}$*

$$(6.24) \quad |u(x, t)| \leq CE_0 \left[(1+|x|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1}e^{-\frac{|x|^2}{M(1+t)}} \right].$$

Proof. Same proof as for Corollary 6.2. □

Lemma 6.19. *For all $t \geq 0$, $x \in \mathbb{R}$, $r > 1$ and all $0 < w < 1$,*

$$(6.25) \quad \int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}}e^{-\frac{|x-wy|^2}{M(1+t)}}(1+|y|)^{-r}dy \leq C \left[(1+|x|+\sqrt{t})^{-r} + (1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{M'(1+t)}} \right],$$

for some sufficiently large $M' > M$.

Proof. We first consider the case of $|x| \leq \sqrt{1+t}$ which implies $e^{-\frac{|x|^2}{M(1+t)}} > e^{-\frac{1}{M}}$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}}e^{-\frac{|x-wy|^2}{M(1+t)}}(1+|y|)^{1-r}dy &\leq (1+t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (1+|y|)^{1-r}dy \\
&\leq C(1+t)^{-\frac{1}{2}} \\
&\leq C(1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{M(1+t)}}.
\end{aligned}$$

For the case of $|x| > \sqrt{1+t}$, we break the integration into two parts.

$$\begin{aligned}
\int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}}e^{-\frac{|x-wy|^2}{M(1+t)}}(1+|y|)^{-r}dy &= \int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{M(1+t)}} \left(1 + \frac{|y|}{w}\right)^{-r} \frac{1}{w}dy \\
&= \int_0^{|x|/2} + \int_{|x|/2}^{\infty} = I + II.
\end{aligned}$$

For part I , we have

$$I \leq (1+t)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{4M(1+t)}} \int_0^{|x|/2} \left(1 + \frac{|y|}{w}\right)^{-r} \frac{1}{w}dy \leq C(1+t)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{4M(1+t)}}.$$

For part II , we have

$$II \leq C(1 + \frac{|x|}{w})^{-r} \frac{1}{w} \int_{|x|/2}^{\infty} (1+t)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{M(1+t)}} dy \leq C(1 + \frac{|x|}{w})^{-r} \frac{1}{w}.$$

Define a function

$$f(w) = \left(1 + \frac{|x|}{(r-1)w}\right)^{-r} \frac{1}{w}.$$

We easily show that $f(1) = \left(1 + \frac{|x|}{r-1}\right)^{-r}$ and $f(w)$ is increasing for $|x| > 1$ which implies that if $|x| > \sqrt{1+t} > 1$, for all $0 < w < 1$, we have

$$II \leq Cf(w) \leq Cf(1) \leq C(1 + |x|)^{-r}.$$

□

Lemma 6.20. For all $t > s > 0$, $x \in \mathbb{R}$, $r > 1$ and all $0 < w < 1$,

$$(6.26) \quad \int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy|^2}{M(1+t-s)}} (1+|y|+\sqrt{s})^{-r} dy \\ \leq C \left[(1+|x|+\sqrt{t-s}+\sqrt{s})^{-r} + (1+t-s)^{-\frac{1}{2}} (1+s)^{-\frac{(r-1)}{2}} e^{-\frac{|x|^2}{M'(1+t)}} \right]$$

for some sufficiently large $M' > M$.

Proof. We consider first the case of $|x| \leq \sqrt{1+t}$ which implies $e^{-\frac{|x|^2}{M(1+t)}} \geq e^{-\frac{1}{M}}$, and so

$$\int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy|^2}{M(1+t-s)}} (1+|y|+\sqrt{s})^{-r} dy \leq (1+t-s)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (1+|y|+\sqrt{s})^{-r} dy \\ \leq C(1+t-s)^{-\frac{1}{2}} (1+\sqrt{s})^{-r+1} \\ \leq C(1+t-s)^{-\frac{1}{2}} (1+\sqrt{s})^{-r+1} e^{-\frac{|x|^2}{M(1+t)}}.$$

For the case of $|x| > \sqrt{1+t}$, we separate the integration into two parts.

$$\int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy|^2}{M(1+t-s)}} (1+|y|+\sqrt{s})^{-r} dy \\ = \int_{-\infty}^{|x|/2} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy|^2}{M(1+t-s)}} \left(1 + \frac{|y|}{w} + \sqrt{s}\right)^{-r} \frac{1}{w} dy \\ = \int_0^{|x|/2} + \int_{|x|/2}^{\infty} = I + II.$$

For I , we have

$$\begin{aligned} I &\leq (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x|^2}{4M(1+t-s)}} \int_0^{|x|/2} \left(1 + \frac{|y|}{w} + \sqrt{s}\right)^{-r} \frac{1}{w} dy \\ &\leq C(1+t-s)^{-\frac{1}{2}} (1+\sqrt{s})^{-r+1} e^{-\frac{|x|^2}{M'(1+t)}}. \end{aligned}$$

For II , we have

$$II \leq C(1 + \frac{|x|}{w} + \sqrt{s})^{-r} \frac{1}{w} \int_{|x|/2}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{M(1+t-s)}} dy \leq C(1 + \frac{|x|}{w} + \sqrt{s})^{-r} \frac{1}{w}.$$

Since $|x| > \sqrt{1+t} > \sqrt{t-s}$,

$$II \leq C(1 + \frac{2|x|}{w} + \sqrt{s})^{-r} \frac{1}{w} \leq C(1 + \frac{|x| + \sqrt{t-s}}{w} + \sqrt{s})^{-r} \frac{1}{w}.$$

Define a function

$$f(w) = \left(1 + \frac{2(|x| + \sqrt{t-s})}{(r-1)w} + \sqrt{s}\right)^{-r} \frac{1}{w}.$$

Then $f(1) = \left(1 + \frac{2(|x| + \sqrt{t-s})}{(r-1)} + \sqrt{s}\right)^{-r}$ and f is increasing. Indeed,

$$f'(w) = \left(1 + \frac{2(|x| + \sqrt{t-s})}{(r-1)w} + \sqrt{s}\right)^{-r-1} \frac{1}{w^3} [(2(|x| + \sqrt{t-s}) - w(1 + \sqrt{s}))]$$

Since $|x| > \sqrt{1+t}$, $|x| > 1$ and $|x| > \sqrt{s}$, that is, $f'(w) > 0$. Thus if $|x| > \sqrt{1+t}$, for all $0 < w < 1$, we have

$$II \leq Cf(w) \leq Cf(1) \leq C(1 + |x| + \sqrt{t-s} + \sqrt{s})^{-r}.$$

□

Lemma 6.21. Suppose $u(x, t)$ satisfies that $u_t = u_{xx}$ and $|u_0(x)| \leq E_0(1+|x|)^{-r}$, $r > 2$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geq 4$. Then for some sufficiently large $M' > M$,

$$(6.27) \quad |u(x, t) - U_0 k(x, t)| \leq CE_0 \left[(1+t)^{-\frac{1}{2}} (1 + |x| + \sqrt{t})^{-r+1} + (1+t)^{-1} e^{-\frac{|x|^2}{M'(1+t)}} \right].$$

where $U_0 = \int_{-\infty}^{\infty} u_0(y) dy$ and $k(x, t) = (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}}$.

Proof. By the Mean Value Theorem, (6.25) and $r - 1 > 1$, we have

$$\begin{aligned}
& |u(x, t) - U_0 k(x, t)| \\
& \leq \int_{-\infty}^{\infty} \int_0^1 |k_x(x - wy, t)| |y| (1 + |y|)^{-r} dw dy \\
& \leq CE_0 (1 + t)^{-\frac{1}{2}} \int_0^1 \int_{-\infty}^{\infty} (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - wy|^2}{M(1+t)}} (1 + |y|)^{-r+1} dy dw \\
& \leq CE_0 \left[(1 + t)^{-\frac{1}{2}} (1 + |x| + \sqrt{t})^{-r+1} + (1 + t)^{-1} e^{-\frac{|x|^2}{M'(1+t)}} \right].
\end{aligned}$$

□

Lemma 6.22. Suppose $u(x, t)$ satisfies that $u_t = u_{xx} + u^q$ and $|u_0(x)| \leq E_0(1 + |x|)^{-r}$, $r > 2$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geq 4$. Then for some sufficiently large $M'' > M' > M$,

$$\begin{aligned}
(6.28) \quad & \left| \int_{-\infty}^{\infty} k(x - y, t - s) u^q(y, s) ds - U(s) k(x, t - s) \right| \\
& \leq CE_0 (1 + s)^{-1} \left[(1 + t - s)^{-\frac{1}{2}} (1 + |y| + \sqrt{t - s} + \sqrt{s})^{-2r+1} + (1 + t - s)^{-1} e^{-\frac{|x|^2}{M''(1+t)}} \right],
\end{aligned}$$

where $U(s) = \int_{-\infty}^{\infty} u^q(y, s) dy$ and $k(x, t) = (1 + t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}}$.

Proof. Noting, by (6.24) and $q \geq 4$, that

$$|yu^q(y, s)| = |u^{p-2}| |yu^2(y, s)| \leq CE_0 (1 + s)^{-1} \left[(1 + |y| + \sqrt{s})^{-2r+1} + (1 + s)^{-\frac{1}{2}} e^{-\frac{|y|^2}{M(1+s)}} \right].$$

we obtain, by Mean Value Theorem again and by (6.26),

$$\begin{aligned}
& \left| \int_{-\infty}^{\infty} k(x - y, t - s) u^q(y, s) ds - U(s) k(x, t - s) \right| \\
& \leq \int_0^1 \int_{-\infty}^{\infty} (1 + t - s)^{-1} e^{-\frac{|x - wy|^2}{M(1+t-s)}} |yu^q(y, s)| dy dw \\
& \leq CE_0 \int_0^1 \int_{-\infty}^{\infty} (1 + t - s)^{-1} e^{-\frac{|x - wy|^2}{M(1+t-s)}} (1 + s)^{-1} (1 + |y| + \sqrt{s})^{-2r+1} dy dw \\
& \quad + CE_0 \int_0^1 \int_{-\infty}^{\infty} (1 + t - s)^{-1} e^{-\frac{|x - wy|^2}{M(1+t-s)}} (1 + s)^{-\frac{3}{2}} e^{-\frac{|x|^2}{M'(1+s)}} dy dw \\
& \leq CE_0 (1 + t - s)^{-\frac{1}{2}} (1 + s)^{-1} (1 + |x| + \sqrt{t - s} + \sqrt{s})^{-2r+1} \\
& \quad + CE_0 \left[(1 + t - s)^{-1} (1 + s)^{-r} e^{-\frac{|x|^2}{M'(1+t)}} + (1 + t - s)^{-1} (1 + s)^{-1} e^{-\frac{|x|^2}{M''(1+t)}} \right] \\
& \leq CE_0 (1 + s)^{-1} \left[(1 + t - s)^{-\frac{1}{2}} (1 + |y| + \sqrt{t - s} + \sqrt{s})^{-2r+1} + (1 + t - s)^{-1} e^{-\frac{|x|^2}{M''(1+t)}} \right]
\end{aligned}$$

□

Theorem 6.23 (Behavior). *Suppose $u(x, t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(y)| \leq E_0(1 + |x|)^{-r}$, $r > 2$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geq 4$. Set*

$$U_* = \int_0^\infty U(s)ds + U_0 = \int_0^\infty \int_{-\infty}^\infty u^q(y, s)dyds + \int_{-\infty}^\infty u_0(y)dy.$$

Then, $|U_*| < \infty$ and for some sufficiently large $M'' > M' > M$,

$$(6.29) \quad |u(x, t) - U_*k(x, t)| \leq CE_0 \left[(1+t)^{-\frac{1}{2}}(1+|x|+\sqrt{t})^{-r+1} + (1+t)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}(1+\ln(1+t)) \right].$$

Proof. Recalling (6.2) and $q \geq 4$, $|U(s)| \leq CE_0(1+s)^{-\frac{3}{2}}$ and so

$$|U_*| \leq CE_0 \int_0^\infty (1+s)^{-\frac{3}{2}}ds + |u_0|_{L^1} < \infty.$$

Now we break $|u(x, t) - U_*k(x, t)|$ into four parts like (6.11). Then we have

$$(6.30) \quad II \leq CE_0(1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{(1+t)}} \int_t^\infty (1+s)^{-\frac{3}{2}}ds \leq CE_0(1+t)^{-1}e^{-\frac{|x|^2}{(1+t)}}.$$

By (6.28), we have

$$\begin{aligned} III &\leq CE_0 \int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-1}(1+|x|+\sqrt{t-s}+\sqrt{s})^{-2r+1}ds \\ &\quad + CE_0 \int_0^t (1+t-s)^{-1}(1+s)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}ds \\ &\leq CE_0(1+|x|+\sqrt{t})^{-2r+1} \int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-1}ds \\ &\quad + CE_0e^{-\frac{|x|^2}{M''(1+t)}} \int_0^t (1+t-s)^{-1}(1+s)^{-1}ds \\ &\leq CE_0 \left[(1+t)^{-\frac{1}{2}}(1+|x|+\sqrt{t})^{-r+1} + (1+t)^{-1}e^{-\frac{|x|^2}{M''(1+t)}} \ln(1+t) \right]. \end{aligned}$$

Since $|U(s)| \leq CE_0(1+s)^{-\frac{3}{2}}$, IV is exactly the same as (6.20) which is

$$(6.31) \quad IV \leq CE_0(1+t)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}.$$

By (6.27) and (6.30)–(6.31), we obtain (6.29).

□

7 Behavior of perturbations of (1.1)

Let $\tilde{u}(x, t)$ be a solution of the system of reaction-diffusion equations

$$(7.1) \quad u_t = u_{xx} + f(u) + cu_x$$

and let $\bar{u}(x)$ be a stationary solution and define perturbations

$$(7.2) \quad u(x, t) = \tilde{u}(x, t) - \bar{u}(x) \quad \text{and} \quad v(x, t) = \tilde{u}(x + \psi(x, t), t) - \bar{u}(x),$$

for some unknown functions $\psi(x, t) : \mathbb{R}^2 \rightarrow \mathbb{R}$ to be determined later.

Plugging $\bar{u}(x, t) = u(x, t) - \bar{u}(x)$ in (7.1), we have

$$(7.3) \quad u_t = Lu + O(|u|^2),$$

where L is the linear operator of (1.2).

In this section, using $v(x, t)$ and the linearized estimates of L we have done in Section 4, we show the behavior of u satisfying (7.3) similarly as in Section 6 for three cases of initial conditions:

- (1) $|u_0|_{L^1 \cap H^1}, |xu_0|_{L^1} \leq E_0$,
- (2) $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$,
- (3) $|u_0(x)| \leq E_0(1 + |x|)^{-r}$, $r > 1$,

where $E_0 > 0$ sufficiently small and $M > 0$ sufficiently large.

By Theorem 1.2, the Green function $G(x, t; y)$ for the linear equation $u_t = Lu$ satisfies the estimates:

$$\begin{aligned} G(x, t; y) &= \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} \bar{u}'(x) \tilde{q}(y, 0) + O((1+t)^{-1} + t^{-\frac{1}{2}} e^{-\eta t}) e^{-\frac{|x-y-at|^2}{Mt}}, \\ G_y(x, t; y) &= \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} \bar{u}'(x) \tilde{q}(y, 0) + O(t^{-1}) e^{-\frac{|x-y-at|^2}{Mt}}, \end{aligned}$$

for some sufficiently large constant $M > 0$ and $\eta > 0$. First off, let $\chi(t)$ be a smooth cut off function defined for $t \geq 0$ such that $\chi(t) = 0$ for $0 \leq t \leq 1$ and $\chi(t) = 1$ for $t \geq 2$ and define

$$(7.4) \quad E(x, t; y) := \bar{u}'(x) e(x, t; y),$$

where

$$e(x, t; y) = \frac{1}{2\pi\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} \tilde{q}(y, 0) \chi(t).$$

Now we set

$$G(x, t; y) = E(x, t; y) + \tilde{G}(x, t; y) \quad \text{and} \quad G_y(x, t; y) = E(x, t; y) + \tilde{G}_y(x, t; y),$$

where

$$|\tilde{G}(x, t; y)| \leq C(1+t)^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{Mt}} \quad \text{and} \quad |\tilde{G}_y(x, t; y)| \leq C t^{-1} e^{-\frac{|x-y-at|^2}{Mt}}.$$

Lemma 7.1 (Nonlinear perturbation equations, [JZ2]). *For v defined in (7.2), we have*

$$(7.5) \quad (\partial_t - L)v = (\partial_t - L)\bar{u}'(x)\psi + Q + R_x - (\partial_x^2 + \partial_t)S + T,$$

where

$$(7.6) \quad Q := f(v(x, t) + \bar{u}(x)) - f(\bar{u}(x)) - df(\bar{u}(x))v = \mathcal{O}(|v|^2),$$

$$(7.7) \quad R := v\psi_t - v\psi_{xx} + (\bar{u}_x + v_x)\frac{\psi_x^2}{1 + \psi_x},$$

$$(7.8) \quad S := v\psi_x = \mathcal{O}(|v||\psi_x|),$$

and

$$(7.9) \quad T := (f(v + \bar{u}) - f(\bar{u}))\psi_x = \mathcal{O}(|v||\psi_x|).$$

Proof. Direct computation; see [JZ2]. □

7.1 Integral representation and ψ -evolution scheme

We now recall the nonlinear iteration scheme of [JZ2]. Using (7.5) and applying Duhamel's principle and setting

$$(7.10) \quad N(x, t) = (Q + R_x - (\partial_x^2 + \partial_t)S + T)(x, t),$$

we obtain the integral representation

$$v(x, t) = \bar{u}'(x)\psi(x, t) + \int_{-\infty}^{\infty} G(x, t; y)v_0(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x, t-s; y)N(y, s)dyds.$$

for the nonlinear perturbation v . Defining ψ implicitly by

$$(7.11) \quad \psi(x, t) := - \int_{-\infty}^{\infty} e(x, t; y)v_0(y)dy - \int_0^t \int_{-\infty}^{\infty} e(x, t-s; y)N(y, s)dyds,$$

we obtain the integral representation

$$(7.12) \quad v(x, t) = \int_{-\infty}^{\infty} \tilde{G}(x, t; y)v_0(y)dy + \int_0^t \int_{-\infty}^{\infty} \tilde{G}(x, t-s; y)N(y, s)dyds.$$

Differentiating and using $e(x, t; y) = 0$ for $0 < t \leq 1$ we obtain

$$(7.13) \quad \partial_t^k \partial_x^m \psi(x, t) = - \int_{-\infty}^{\infty} \partial_t^k \partial_x^m e(x, t; y)v_0(y)dy - \int_0^t \int_{-\infty}^{\infty} \partial_t^k \partial_x^m e(x, t-s; y)N(y, s)dyds.$$

Together, (7.12) and (7.13) form a complete system in $(v, \partial_t^k \psi, \partial_x^m \psi)$, $0 \leq k \leq 1$, $0 \leq m \leq 2$, that is, v and derivatives of ψ , from solutions of which we may afterward recover the shift function ψ by integration in x , completing the description of \tilde{u} .

7.2 Behavior for initial perturbation $|u_0|_{L^1 \cap H^1}, |xu_0|_{L^1}$ sufficiently small

Theorem 7.2 (Nonlinear stability, [JZ2]). *Let $v(x, t)$ and $u(x, t)$ be defined as in (7.2) and $|u_0(x)| = |v_0(x)|_{L^1 \cap H^1(\mathbb{R})} < E_0$ sufficiently small. Then for all $t \geq 0$ and $p \geq 1$ we have the estimates*

$$(7.14) \quad \begin{aligned} |v(\cdot, t)|_{L^p(\mathbb{R})}(t) &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \\ |u(\cdot, t)|_{L^p(\mathbb{R})}(t), \quad |\psi(\cdot, t)|_{L^p(\mathbb{R})}(t) &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \\ |v(\cdot, t)|_{H^K(\mathbb{R})}(t), \quad |(\psi_t, \psi_x)(\cdot, t)|_{H^K(\mathbb{R})}(t) &\leq CE_0(1+t)^{-\frac{3}{4}}. \end{aligned}$$

(Note: This is proved in [JZ2] for $p \geq 2$. For $p = 1$, we use the integration by part of (7.11) and (7.12) and use $|(Q, R, S, T)|_{L^1} \leq |(v, \psi_x, \psi_t)|_{H^1}^2 \leq CE_0(1+t)^{-\frac{3}{2}}$ to prove $|v(\cdot, t)|_{L^1} \leq CE_0(1+t)^{-\frac{1}{2}}$ and $|\psi(\cdot, t)|_{L^1} \leq CE_0$.)

Lemma 7.3. *For E defined as in (7.4) and $|u_0|_{L^1 \cap H^1}, |xu_0|_{L^1} < E_0$, we have*

$$(7.15) \quad \left| \int_{-\infty}^{\infty} E(x, t; y) u_0(y) dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x, t) \right|_{L^p(x)} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}},$$

where $\bar{U}_0 = \int_{-\infty}^{\infty} u_0(y) \tilde{q}(y, 0) dy$ and $\bar{k}(x, t) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-at|^2}{(4bt)}}$.

Proof. By the Mean Value Theorem,

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} E(x, t; y) u_0(y) dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x, t) \right|_{L^p(x)} \\ &\leq C \int_{-\infty}^{\infty} \int_0^1 |\bar{k}_x(x - wy, t)|_{L^p(x)} |y u_0(y)| dw dy \\ &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \end{aligned}$$

□

Lemma 7.4. *Associated with the solution $(u, \psi_t, \psi_x, \psi_{xx})$ of integral system (7.12)–(7.13), we define*

$$(7.16) \quad \zeta(t) := \sup_{0 \leq s \leq t} |(x - as)(v, \psi_t, \psi_x, \psi_{xx})|_{L^1(x)}(s)$$

Then for all $t \geq 0$ for which $\zeta(t)$ is sufficiently small, we have the estimate

$$(7.17) \quad \zeta(t) \leq C(E_0 + \zeta^2(t))$$

for some constant $C > 0$, as long as $|v_0|_{L^1 \cap H^1}, |xv_0|_{L^1} < E_0$, for $E_0 > 0$ sufficiently small.

Proof. To begin, notice first that

$$\begin{aligned}
& |(y - as)(Q + T + R + S)(y, s)|_{L^1(y)} \\
& \leq |(y - as)(v^2 + \psi_t^2 + \psi_y^2 + \psi_{yy}^2)|_{L^1(y)} \\
& \leq (|v|_{L^\infty} + |\psi_t|_{L^\infty} + |\psi_x|_{L^\infty} + |\psi_{xx}|_{L^\infty})|(x - as)(v, \psi_t, \psi_x, \psi_{xx})|_{L^1(x)} \\
& \leq CE_0(1 + t)^{-1}\zeta(t),
\end{aligned}$$

and

$$(7.18) \quad |(Q + T + R + S)(y, s)|_{L^1(y)} \leq |(v^2 + \psi_t^2 + \psi_y^2 + \psi_{yy}^2)|_{L^1(y)} \leq (1 + s)^{-\frac{3}{2}}.$$

By the integration by part, we have

$$\begin{aligned}
& |(x - at)v(x, t)|_{L^1(x)} \\
& = \int_{-\infty}^{\infty} \left| (x - at - y)(1 + t)^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{|x-at-y|^2}{Mt}} \right|_{L^1(x)} |v_0(y)| dy \\
& \quad + \int_{-\infty}^{\infty} \left| y(1 + t)^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{|x-at-y|^2}{Mt}} \right|_{L^1(x)} |v_0(y)| dy \\
& \quad + \int_0^t \int_{-\infty}^{\infty} \left| (x - at - (y - as))(1 + t - s)^{-\frac{1}{2}} (t - s)^{-\frac{1}{2}} e^{-\frac{|x-a(t-s)-y|^2}{M(t-s)}} \right|_{L^1(x)} |Q + T| dy ds \\
& \quad + \int_0^t \int_{-\infty}^{\infty} \left| (1 + t - s)^{-\frac{1}{2}} (t - s)^{-\frac{1}{2}} e^{-\frac{|x-a(t-s)-y|^2}{M(t-s)}} \right|_{L^1(x)} |(y - as)(Q + T)| dy ds \\
& \quad + \int_0^t \int_{-\infty}^{\infty} \left| (x - at - (y - as))(t - s)^{-1} e^{-\frac{|x-a(t-s)-y|^2}{M(t-s)}} \right|_{L^1(x)} |R + S| dy ds \\
& \quad + \int_0^t \int_{-\infty}^{\infty} \left| (t - s)^{-1} e^{-\frac{|x-a(t-s)-y|^2}{M(t-s)}} \right|_{L^1(x)} |(y - as)(R + S)| dy ds \\
& \leq |v_0|_{L^1} + (1 + t)^{-\frac{1}{2}} |yv_0|_{L^1} \\
& \quad + \int_0^t |(Q + R + S + T)|_{L^1} ds + \int_0^t (1 + t - s)^{-\frac{1}{2}} |(y - as)(Q + R + S + T)|_{L^1} ds \\
& \leq CE_0 + C(1 + t)^{-\frac{1}{2}} E_0 + CE_0 \int_0^t (1 + s)^{-\frac{3}{2}} ds + CE_0 \zeta(t) \int_0^t (t - s)^{-\frac{1}{2}} (1 + s)^{-1} ds \\
& \leq CE_0 + C(1 + t)^{-\frac{1}{2}} E_0 + CE_0 \zeta(t) \int_0^t (1 + t - s)^{-\frac{1}{2}} (1 + s)^{-1} ds \\
& \leq CE_0 + CE_0(1 + t)^{-\frac{1}{2}} + CE_0 \zeta(t) (1 + t)^{-\frac{1}{2}} \\
& \leq C(E_0 + \zeta^2(t)).
\end{aligned}$$

Similarly, we have

$$|(x - at)(\psi_t, \psi_x, \psi_{xx})|_{L^1(x)} \leq C(E_0 + \zeta^2(t)).$$

□

Corollary 7.5. For $|v_0|_{L^1 \cap H^1}$, $|xv_0|_{L^1} < E_0$, and $E_0 > 0$ sufficiently small,

$$(7.19) \quad |(y - as)(Q + T + R + S)(y, s)|_{L^1(y)} \leq CE_0(1 + s)^{-1}.$$

Lemma 7.6. Recalling (7.4) and (7.10), we have

$$(7.20) \quad \left| \int_{-\infty}^{\infty} E(x, t - s; y) N(y, s) dy - \bar{U}(s) \bar{u}'(x) \bar{k}(x - as, t - s) \right|_{L^p(x)} \\ \leq C(1 + t - s)^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2}} (1 + s)^{-1},$$

where $\bar{U}(s) = \int_{-\infty}^{\infty} N(y, s) \tilde{q}(y, 0) dy$.

Proof. By integration by parts, the Mean Value Theorem and (7.18)–(7.19), we have

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} E(x, t - s; y) N(y, s) dy - \bar{U}(s) \bar{u}'(x) \bar{k}(x - as, t - s) \right|_{L^p(x)} \\ & \leq \int_{-\infty}^{\infty} |\bar{u}'(x) \tilde{q}(y, 0)| |\bar{k}(x - y, t - s) - \bar{k}(x - as, t - s)|_{L^p(x)} |(Q + T)(y, s)| dy \\ & \quad + \left| \int_{-\infty}^{\infty} \bar{u}'(x) \tilde{q}_y(y, 0) (\bar{k}(x - y, t - s) - \bar{k}(x - as, t - s)) (R + S)(y, s) dy \right|_{L^p(x)} \\ & \quad + \left| \int_{-\infty}^{\infty} \bar{u}'(x) \tilde{q}(y, 0) \partial_y (\bar{k}(x - y, t - s) - \bar{k}(x - as, t - s)) (R + S)(y, s) dy \right|_{L^p(x)} \\ & \leq CE_0(1 + t - s)^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2}} |(y - as)(Q + T + R + S)(y, s)|_{L^1(y)} \\ & \quad + CE_0(1 + t - s)^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2}} |(Q + T + R + S)(y, s)|_{L^1(y)} \\ & \leq +CE_0(1 + t - s)^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2}} (1 + s)^{-1}. \end{aligned}$$

□

Theorem 7.7 (Behavior). Suppose $u(x, t)$ satisfies $u_t = Lu + O(|u|^2)$ and $|u_0|_{L^1 \cap H^1}$, $|xu_0|_{L^1} < E_0$, with $E_0 > 0$ sufficiently small. Set

$$\bar{U}_* = \int_0^\infty \bar{U}(s) ds + \bar{U}_0,$$

Then $|\bar{U}_*| < \infty$ and

$$(7.21) \quad |u(x, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)} \leq CE_0(1 + t)^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2}} (1 + \ln(1 + t)).$$

Proof. Noting first, by integration by part and (7.14), that

$$(7.22) \quad |\bar{U}(s)| \leq C|(Q, R, S, T)(y, s)|_{L^1(y)} \leq |(v, \psi_t, \psi_x)|_{H^1}^2 \leq CE_0(1 + s)^{-\frac{3}{2}},$$

we have $|\bar{U}_*| \leq C \int_0^\infty (1+s)^{-\frac{3}{2}} ds + CE_0|u_0|_{L^1} < \infty$.

Now we break $|u(x, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)}$ into three parts.

$$\begin{aligned}
(7.23) \quad & |u(x, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)} \\
&= |\tilde{u}(x, t) - \bar{u}(x) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)} \\
&= |\tilde{u}(x + \psi, t) - \bar{u}(x) + \tilde{u}(x, t) - \tilde{u}(x + \psi, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)} \\
&\leq |v(x, t)|_{L^p(x)} + |\tilde{u}(x, t) - \tilde{u}(x + \psi, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)} \\
&= |v(x, t)|_{L^p(x)} + |\tilde{u}_x(x + \psi, t)(1 + \psi_x)\psi + O(|\psi|^2) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)} \\
&= |v(x, t)|_{L^p(x)} + |(\bar{u}'(x) + v_x)\psi + O(|\psi|^2) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)} \\
&\leq |v(x, t)|_{L^p(x)} + (|v_x||\psi| + O(|\psi|^2))_{L^p(x)} + |\bar{u}'(x)\psi - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)}.
\end{aligned}$$

By (7.14), we easily see first two terms

$$\begin{aligned}
(7.24) \quad & |v(x, t)|_{L^p(x)} + (|v_x||\psi| + |\psi|^2)_{L^p(x)} \\
&\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} + |v_x|_{L^\infty}|\psi|_{L^p} + |\psi|_{L^{2p}}^2 \\
&\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} + C(1+t)^{-\frac{3}{4}}(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} + C(1+t)^{-(1-\frac{1}{2p})} \\
&\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.
\end{aligned}$$

Now we estimate the last term

$$\begin{aligned}
(7.25) \quad & |\bar{u}'(x)\psi - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|_{L^p(x)} \\
&= \left| \int_{-\infty}^\infty E(x, t; y) u_0(y) dy + \int_0^t \int_{-\infty}^\infty E(x, t-s; y) N(y, s) dy ds - \bar{U}_* \bar{u}'(x) \bar{k}(x, t) \right|_{L^p(x)} \\
&\leq \left| \int_{-\infty}^\infty E(x, t; y) u_0(y) dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x, t) \right|_{L^p(x)} \\
&\quad + \int_t^\infty \left| \bar{u}'(x) \bar{k}(x, t) \bar{U}(s) \right|_{L^p(x)} ds \\
&\quad + \int_0^t \left| \int_{-\infty}^\infty E(x, t-s; y) N(y, s) dy - \bar{U}(s) \bar{u}'(x) \bar{k}(x-as, t-s) \right|_{L^p(x)} ds \\
&\quad + \int_0^t |\bar{U}(s) \bar{u}'(x)| |\bar{k}(x-as, t-s) - \bar{k}(x, t)|_{L^p(x)} ds \\
&= I + II + III + IV.
\end{aligned}$$

Since $|\bar{U}(s)| \leq CE_0(1+s)^{-\frac{3}{2}}$,

$$(7.26) \quad II \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \int_t^\infty (1+s)^{-\frac{3}{2}} ds \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$

By (7.20), we have

$$(7.27) \quad III \leq C \int_0^t (1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} (1+s)^{-1} ds \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} (1+\ln(1+t)).$$

By the Mean Value Theorem, for some $s^* \in (0, t/2)$, we have

$$(7.28) \quad \begin{aligned} IV &\leq C \int_{t/2}^t (1+s)^{-\frac{3}{2}} |\bar{k}(x-as, t-s) - \bar{k}(x, t)|_{L^p(x)} ds \\ &\quad + C \int_0^{t/2} (1+s)^{-\frac{3}{2}} s |\bar{k}_t(x-as, t-s^*)|_{L^p(x)} ds \\ &\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \int_{t/2}^t (1+s)^{-\frac{3}{2}} ds + C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1} \int_0^{t/2} (1+s)^{-\frac{1}{2}} ds \\ &\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \end{aligned}$$

By (7.15) and (7.24)–(7.28), we obtain the result (7.21). \square

Remark 7.8. *Untangling coordinate changes, we see that $\bar{U}_* \bar{k}(x, t)$ is an estimate for $\psi(x, t)$; that is, $|\bar{u}(x) - \bar{u}(x - \bar{U}_* \bar{k}(x, t))| \sim |\bar{U}_* \bar{u}' \bar{k}|$. This makes a connection between the analyses of [JZ2] (where v and ψ but not $\bar{U}_* \bar{k}(x, t)$ appear) and [S1, S2] (where the equivalent of $\bar{U}_* \bar{k}(x, t)$ appears, but not v or ψ).*

7.3 Behavior for initial perturbation $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$

To show behavior of u , we first consider pointwise bounds of v , ψ_t , ψ_x and ψ_{xx} like previous one. In this section, we take $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large.

Lemma 7.9. *Suppose $|v_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. For v , ψ_t , ψ_x and ψ_{xx} defined in (7.12) and (7.13), define*

$$(7.29) \quad \zeta(t) := \sup_{0 \leq s \leq t, x \in \mathbb{R}} |(v, \psi_t, \psi_x, \psi_{xx})|(1+s) e^{\frac{|x-as|^2}{M(1+s)}}.$$

Then, for all $t \geq 0$ for which $\zeta(t)$ defined in (7.29) is finite,

$$(7.30) \quad \zeta(t) \leq C(E_0 + \zeta(t)^2)$$

for some constant $C > 0$.

Proof. Note first that by (7.14), we have $|v_x|_\infty \leq |v|_{H^1} \leq CE_0(1+t)^{-\frac{3}{4}} \leq C$ and so by (7.6) – (7.9) and (7.29) we have

$$|(Q, R, S, T)(x, t)| \leq |(v, \psi_t, \psi_x, \psi_{xx})(x, t)|^2 \leq \zeta(t)^2 (1+t)^{-2} e^{-\frac{|x-at|^2}{M(1+t)}}.$$

Thus, from (7.12), we have

(7.31)

$$\begin{aligned}
|v(x, t)| &\leq \int_{-\infty}^{\infty} |\tilde{G}(x, t; y)| |v_0(y)| dy + \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x, t-s; y)| |(Q, R, S, T)(y, s)| ds dy \\
&\leq CE_0 \int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{Mt}} e^{-\frac{|y|^2}{M}} dy \\
&\quad + C\zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1} e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}} (1+s)^{-2} e^{-\frac{|y-as|^2}{M(1+s)}} dy ds \\
&\leq CE_0 t^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} + C\zeta^2(t) (1+t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M(1+t)}} \int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{3}{2}} ds \\
&\leq C(E_0 + \zeta^2(t)) (1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}},
\end{aligned}$$

here we use the integration by parts to exchange the ∂_y and $(\partial_y^2 + \partial_s)$ derivatives on R and S respectively for $-\partial_y$ and $(\partial_y^2 - \partial_s)$ derivatives on \tilde{G} and recall $|\tilde{G}_{yy} + \tilde{G}_t| \sim |\tilde{G}_y| \leq C t^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M(1+t)}}$.

Recalling $e(x, t; y) = 0$ for $0 < t \leq 1$ and from (7.13), we have

$$\begin{aligned}
&|(\psi_t, \psi_x, \psi_{xx})(x, t)| \\
&\leq \int_{-\infty}^{\infty} |e_x(x, t; y)| |v_0(y)| dy + \int_0^t \int_{-\infty}^{\infty} |e_x(x, t-s; y)| |(Q, R, S, T)(y, s)| ds dy \\
&\leq E_0 \int_{-\infty}^{\infty} (1+t)^{-1} e^{-\frac{|x-y-at|^2}{Mt}} e^{-\frac{|y|^2}{M}} dy \\
&\quad + \zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (1+t-s)^{-1} e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}} (1+s)^{-2} e^{-\frac{|y-as|^2}{M(1+s)}} dy ds \\
&\leq CE_0 (1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} + C\zeta^2(t) (1+t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M(1+t)}} \int_0^t (1+t-s)^{-\frac{1}{2}} (1+s)^{-\frac{3}{2}} ds \\
&\leq C(E_0 + \zeta^2(t)) (1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}.
\end{aligned}
\tag{7.32}$$

The (7.31) and (7.32) implies (7.30). □

Corollary 7.10. *For v defined in (7.2) with $|v_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large,*

$$|v(x, t)| \leq CE_0 (1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}. \tag{7.33}$$

Proof. Same proof as Corollary 6.2. □

Lemma 7.11. *Let E be defined as in (7.4) and $|u_0(x)| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. Then, for some sufficiently large $M' > M$,*

$$(7.34) \quad \left| \int_{-\infty}^{\infty} E(x, t; y) u_0(y) dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x, t) \right| \leq C E_0 (1+t)^{-1} e^{-\frac{|x-at|^2}{M'(1+t)}},$$

where $\bar{U}_0 = \int_{-\infty}^{\infty} u_0(y) \tilde{q}(y, 0) dy$ and $\bar{k}(x, t) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-at|^2}{(4bt)}}$.

Proof. By the Mean Value Theorem,

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} E(x, t; y) u_0(y) dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x, t) \right| \\ & \leq C \int_{-\infty}^{\infty} |\bar{k}(x-y, t) - \bar{k}(x, t)| |u_0(y)| dy \\ & \leq C E_0 \int_{-\infty}^{\infty} \int_0^1 (1+t)^{-1} e^{-\frac{|x-wy-at|^2}{(1+t)}} e^{-\frac{|y|^2}{M}} dw dy \\ & \leq C E_0 (1+t)^{-1} e^{-\frac{|x-at|^2}{M'(1+t)}}. \end{aligned}$$

□

Lemma 7.12. *Recalling (7.4) and (7.10), we have for some sufficiently large $M'' > M' > M$,*

$$(7.35) \quad \left| \int_{-\infty}^{\infty} E(x, t-s; y) N(y, s) dy - \bar{U}(s) \bar{u}'(x) \bar{k}(x-as, t-s) \right| \leq C E_0 (1+t-s)^{-1} (1+s)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}},$$

where $\bar{U}(s) = \int_{-\infty}^{\infty} N(y, s) \tilde{q}(y, 0) dy$.

Proof. Noting first that $|(Q, R, S, T)| \leq CE_0(1+t)^{-2}e^{-\frac{|x-at|^2}{M(1+t)}}$, we have

$$\begin{aligned}
& \left| \int_{-\infty}^{\infty} E(x, t-s; y) N(y, s) dy - \bar{U}(s) \bar{u}'(x) \bar{k}(x, t-s) \right| \\
&= \left| \int_{-\infty}^{\infty} \bar{k}(x-y, t-s) \tilde{q}(y, 0) \bar{u}'(x) N(y, s) dy - \int_{-\infty}^{\infty} N(y, s) \tilde{q}(y, 0) \bar{u}'(x) \bar{k}(x, t-s) dy \right| \\
&= \left| \int_{-\infty}^{\infty} \bar{u}'(x) N(y, s) \tilde{q}(y, 0) (\bar{k}(x-y, t-s) - \bar{k}(x-as, t-s)) dy \right| \\
&\leq \int_{-\infty}^{\infty} |\bar{u}'(x) \tilde{q}(y, 0)| |\bar{k}(x-y, t-s) - \bar{k}(x-as, t-s)| |(Q+T)(y, s)| dy \\
&\quad + \left| \int_{-\infty}^{\infty} \bar{u}'(x) \tilde{q}_y(y, 0) (\bar{k}(x-y, t-s) - \bar{k}(x-as, t-s)) (R+S)(y, s) dy \right| \\
&\quad + \left| \int_{-\infty}^{\infty} \bar{u}'(x) \tilde{q}(y, 0) \partial_y (\bar{k}(x-y, t-s) - \bar{k}(x-as, t-s)) (R+S)(y, s) dy \right| \\
&\leq CE_0 \int_{-\infty}^{\infty} \int_0^1 (1+t-s)^{-1} e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}} |y-as| (1+s)^{-2} e^{-\frac{|y-as|^2}{M(1+s)}} dw dy \\
&\quad + CE_0 \int_{-\infty}^{\infty} (1+t-s)^{-1} e^{-\frac{|x-y-a(t-s)|^2}{(t-s)}} (1+s)^{-2} e^{-\frac{|y-as|^2}{M(1+s)}} dy \\
&\leq CE_0 \int_{-\infty}^{\infty} \int_0^1 (1+t-s)^{-1} e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}} (1+s)^{-\frac{3}{2}} e^{-\frac{|y-as|^2}{M'(1+s)}} dw dy \\
&\leq CE_0 (1+t-s)^{-1} (1+s)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}}.
\end{aligned}$$

□

Theorem 7.13 (Behavior). *Suppose $u(x, t)$ satisfies $u_t = Lu + O(|u|^2)$ and $|u_0| \leq E_0 e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. Set*

$$\bar{U}_* = \int_0^\infty \bar{U}(s) ds + \bar{U}_0,$$

Then $|\bar{U}_| < \infty$ and for some sufficiently large $M'' > M' > M$,*

$$(7.36) \quad |u(x, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)| \leq C(1+t)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}} (1 + \ln(1+t)).$$

Proof. Recalling $|\bar{U}(s)| \leq CE_0(1+s)^{-\frac{3}{2}}$, we have $|\bar{U}_*| < \infty$. We first break $|u(x, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|$ into three parts exactly the same as (7.23). By (7.33) and (7.11), we easily see first two terms

$$(7.37) \quad |v(x, t)| + O(|v_x| |\psi| + |\psi|^2) \leq C(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}.$$

Now we break the last term into four parts exactly the same as (7.25). Then

$$(7.38) \quad II \leq CE_0(1+t)^{-\frac{1}{2}}e^{-\frac{|x-at|^2}{M(1+t)}} \int_t^\infty (1+s)^{-\frac{3}{2}}ds \leq CE_0(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}.$$

By (7.35), we have

$$(7.39) \quad III \leq CE_0 \int_0^t (1+t-s)^{-1}(1+s)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}ds \leq CE_0(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}} \ln(1+t).$$

By the Mean Value Theorem, for some $s^* \in (0, t/2)$, we have

$$(7.40) \quad \begin{aligned} IV &\leq CE_0 \left[\int_{t/2}^t (1+s)^{-\frac{3}{2}} |\bar{k}(x-as, t-s) - \bar{k}(x, t)| ds + \int_0^{t/2} (1+s)^{-\frac{3}{2}} s |\bar{k}_t(x-as, t-s^*)| ds \right] \\ &\leq CE_0(1+t)^{-\frac{1}{2}}e^{-\frac{|x-at|^2}{M(1+t)}} \int_{t/2}^t (1+s)^{-\frac{3}{2}} ds + CE_0e^{-\frac{|x-at|^2}{M(1+t)}} \int_0^{t/2} (1+s)^{-\frac{1}{2}}(1+t-s)^{-\frac{3}{2}} ds \\ &\leq CE_0(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}. \end{aligned}$$

By (7.34) and (7.37)–(7.40), we obtain the result (7.36). \square

7.4 Behavior for initial perturbation $|v_0(x)| \leq E_0(1+|x|)^{-r}$

The proof of following lemma and corollary are exactly the same as (6.21) and (6.22) replacing $|x|$ by $|x-at|$. In this section, we take $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large.

Lemma 7.14. *For all $t \geq 0$ and $r > 1$, and any $x \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{t}} (1+|y|)^{-r} dy \leq C \left[t^{-\frac{1}{2}} \wedge (1+|x-at|)^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{Mt}} \right],$$

for some sufficiently large $M > 0$ and $C > 0$.

Corollary 7.15. *For all $t \geq 0$ and $r > 1$, and any $x \in \mathbb{R}$,*

$$(7.41) \quad \int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{t}} (1+|y|)^{-r} dy \leq C \left[(1+|x-at|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{Mt}} \right],$$

for some $M > 0$ sufficiently large and $C > 0$.

Lemma 7.16. *Suppose $|v_0(x)| \leq E_0(1+|x|)^{-r}$, $r > 1$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. For v, ψ_t, ψ_x and ψ_{xx} defined in (7.12) and (7.13), define*

$$(7.42) \quad \zeta(t) := \sup_{0 \leq s \leq t, x \in \mathbb{R}} |(v, \psi_t, \psi_x, \psi_{xx})|(1+s)^{\frac{1}{2}} \left[(1+|x-as|+\sqrt{s})^{-r} + (1+\sqrt{s})^{-1} e^{-\frac{|x-as|^2}{M(1+s)}} \right]^{-1}.$$

Then, for all $t \geq 0$ for which $\zeta(t)$ is finite, we have

$$(7.43) \quad \zeta(t) \leq C(E_0 + \zeta(t)^2)$$

for some constant $C > 0$.

Proof. Note first that by (7.14), we have $|v_x|_\infty \leq |v|_{H^1} \leq CE_0(1+t)^{-\frac{3}{4}} \leq C$ and so by (7.6) – (7.9) and (7.42) we have

$$\begin{aligned} |(Q, R, S, T)(x, t)| &\leq |(v, \psi_t, \psi_x, \psi_{xx})(x, t)|^2 \\ &\leq \zeta(t)^2(1+s)^{-1}[(1+|x-as|+\sqrt{s})^{-r} + (1+\sqrt{s})^{-1}e^{-\frac{|x-as|^2}{M(1+s)}}]^2 \end{aligned}$$

Then, from (7.12), we have

$$\begin{aligned} |v(x, t)| &\leq \int_{-\infty}^{\infty} |\tilde{G}(x, t; y)| |v_0(y)| dy + \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x, t-s; y)| |(Q, R, S, T)(y, s)| ds dy \\ &\leq CE_0 \int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{Mt}} (1+|y|)^{-r} dy \\ &\quad + C\zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (1+s)^{-1} (t-s)^{-1} e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}} (1+|y-as|+\sqrt{s})^{-2r} dy ds \\ &\quad + C\zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (1+s)^{-1} (t-s)^{-1} e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}} (1+\sqrt{s})^{-2} e^{-\frac{|y-as|^2}{M(1+s)}} dy ds \\ &= I + II + III. \end{aligned}$$

By (7.41), we have

$$I \leq CE_0(1+t)^{-\frac{1}{2}} \left[(1+|x-at|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{Mt}} \right],$$

For *III*, we have

$$\begin{aligned} III &= \zeta^2(t) \int_0^t (1+s)^{-2} \int_{-\infty}^{\infty} (t-s)^{-1} e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}} e^{-\frac{|y-as|^2}{M(1+s)}} dy ds \\ &= \zeta^2(t) (1+t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M(1+t)}} \int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{3}{2}} ds \\ &\leq \zeta^2(t) (1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}. \end{aligned}$$

For II , by (7.41), we estimate

$$\begin{aligned}
II &\leq C\zeta^2(t) \int_0^t (1+s)^{-(1+\frac{r}{2})} (t-s)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-(y-as)-at|^2}{M(t-s)}} (1+|y-as|)^{-r} dy ds \\
&\leq C\zeta^2(t) \int_0^t (1+s)^{-\frac{3}{2}} (t-s)^{-\frac{1}{2}} (1+|x-at|+\sqrt{t-s})^{-r} ds \\
&\quad + C\zeta^2(t) \int_0^t (1+s)^{-(1+\frac{r}{2})} (t-s)^{-\frac{1}{2}} (1+\sqrt{t-s})^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} ds \\
&\leq C\zeta^2(t) (1+t)^{-\frac{1}{2}} [(1+|x-at|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}].
\end{aligned}$$

Now we consider $|(\psi_t, \psi_x, \psi_{xx})|$. Recalling $e(x, t; y) = 0$ for $0 < t \leq 1$, similarly we have

$$\begin{aligned}
&|(\psi_t, \psi_x, \psi_{xx})(x, t)| \\
&\leq \int_{-\infty}^{\infty} |e_x(x, t; y)| |v_0(y)| dy + \int_0^t \int_{-\infty}^{\infty} |e_x(x, t-s; y)| |(Q, R, S, T)(y, s)| ds dy \\
&\leq E_0 \int_{-\infty}^{\infty} (1+t)^{-1} e^{-\frac{|x-y-at|^2}{Mt}} (1+|y|)^{-r} dy \\
&\quad + C\zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (1+s)^{-1} (1+t-s)^{-1} e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}} (1+|y-as|+\sqrt{s})^{-2r} dy ds \\
&\quad + C\zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (1+s)^{-1} (1+t-s)^{-1} e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}} (1+\sqrt{s})^{-2} e^{-\frac{|y-as|^2}{M(1+s)}} dy ds \\
&\leq C\zeta^2(t) (1+t)^{-\frac{1}{2}} [(1+|x-at|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}].
\end{aligned}$$

□

Corollary 7.17. *For v defined in (7.2) with $|v_0(x)| \leq E_0(1+|x|)^{-r}$, $r > 1$, $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large,*

$$(7.44) \quad |v(x, t)| \leq CE_0(1+t)^{-\frac{1}{2}} [(1+|x-at|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}]$$

Proof. Same proof as Corollary 6.2. □

The proofs of (7.45) and (7.46) in the following lemmas are the same as those for (6.25) and (6.26), respectively.

Lemma 7.18. *For all $t > 0$, $x \in \mathbb{R}$, $r > 2$ and all $0 < w < 1$,*

$$(7.45) \quad \int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}} e^{-\frac{|x-wy-at|^2}{M(1+t)}} (1+|y|)^{-r} dy \leq CE_0 [(1+|x-at|+\sqrt{t})^{-r} + (1+t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M'(1+t)}}].$$

for some sufficiently large $M' > M$.

Lemma 7.19. *For all $t > s > 0$, $x \in \mathbb{R}$, $r > 2$ and all $0 < w < 1$,*

$$(7.46) \quad \int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy-at|^2}{M(1+t-s)}} (1+|y|+\sqrt{s})^{-r} dy \\ \leq CE_0 \left[(1+|x-at|+\sqrt{t-s}+\sqrt{t})^{-r} + (1+t-s)^{-\frac{1}{2}} (1+s)^{-\frac{r}{2}} e^{-\frac{|x-at|^2}{M'(1+t)}} \right].$$

for some sufficiently large $M' > M$.

Lemma 7.20. *Suppose $u(x, t)$ satisfies $u_t = Lu$ and $|u_0(x)| \leq E_0(1+|x|)^{-r}$, $r > 2$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. Then for some sufficiently large $M' > M$,*

$$(7.47) \quad \left| \int_{-\infty}^{\infty} E(x, t; y) u_0(y) dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x, t) \right| \\ \leq CE_0 \left[(1+t)^{-\frac{1}{2}} (1+|x-at|+\sqrt{t})^{-r+1} + (1+t)^{-1} e^{-|x-at|^2/M'(1+t)} \right],$$

where $\bar{U}_0 = \int_{-\infty}^{\infty} u_0(y) \tilde{q}(y, 0) dy$ and $\bar{k}(x, t) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-at|^2}{(4bt)}}$.

Proof. By (7.41) and (6.25), we have

$$\left| \int_{-\infty}^{\infty} E(x, t; y) u_0(y) dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x, t) \right| \\ \leq CE_0 \int_{-\infty}^{\infty} \int_0^1 (1+t)^{-1} e^{-\frac{|x-wy-at|^2}{(1+t)}} (1+|y|)^{-r+1} dw dy \\ \leq CE_0 \left[(1+t)^{-\frac{1}{2}} (1+|x-at|+\sqrt{t})^{-r+1} + (1+t)^{-1} e^{-\frac{|x-at|^2}{M'(1+t)}} \right]$$

□

Lemma 7.21. *Recalling (7.4) and (7.10), we have for some sufficiently large $M' > M$*

$$(7.48) \quad \left| \int_{-\infty}^{\infty} E(x, t-s; y) N(y, s) dy - \bar{U}(s) \bar{u}'(x) \bar{k}(x-as, t-s) \right| \\ \leq CE_0 (1+s)^{-1} \left[(1+t-s)^{-\frac{1}{2}} (1+|x-at|+\sqrt{t-s}+\sqrt{s})^{-2r+1} + (1+t-s)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \right].$$

where $\bar{U}(s) = \int_{-\infty}^{\infty} N(y, s) \tilde{q}(y, 0) dy$

Proof. Noting first that

$$|(Q, R, S, T)| \leq CE_0 (1+t)^{-1} \left[(1+|x-at|+\sqrt{t})^{-2r} + (1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \right],$$

we have, from (7.46) and by the Mean Value Theorem,

$$\begin{aligned}
& \left| \int_{-\infty}^{\infty} E(x, t-s; y) N(y, s) dy - \bar{U}(s) \bar{u}'(x) \bar{k}(x, t-s) \right| \\
& \leq C \int_{-\infty}^{\infty} |\bar{k}(x-y, t-s) - \bar{k}(x, t-s)| |(Q, R, S, T)(y, s)| dy \\
& \leq CE_0 \int_{-\infty}^{\infty} \int_0^1 (1+t-s)^{-1} e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}} |y-as|(1+s)^{-1} (1+|y-as|+\sqrt{s})^{-2r} dw dy \\
& \quad + CE_0 \int_{-\infty}^{\infty} \int_0^1 (1+t-s)^{-1} e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}} |y-as|(1+s)^{-2} e^{-\frac{|y-as|^2}{M(1+s)}} dw dy \\
& \leq CE_0 \int_{-\infty}^{\infty} \int_0^1 (1+t-s)^{-1} e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}} (1+s)^{-1} (1+|y-as|+\sqrt{s})^{-2r+1} dw dy \\
& \quad + CE_0 \int_{-\infty}^{\infty} \int_0^1 (1+t-s)^{-1} e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}} (1+s)^{-\frac{3}{2}} e^{-\frac{|y-as|^2}{M'(1+s)}} dw dy \\
& \leq CE_0 (1+t-s)^{-\frac{1}{2}} (1+s)^{-1} (1+|x-at|+\sqrt{t-s}+\sqrt{t})^{-2r+1} \\
& \quad + CE_0 (1+t-s)^{-1} (1+s)^{-r} e^{-\frac{|x-at|^2}{M(1+t)}} + CE_0 (1+t-s)^{-1} (1+s)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}} \\
& \leq CE_0 (1+s)^{-1} \left[(1+t-s)^{-\frac{1}{2}} (1+|x-at|+\sqrt{t-s}+\sqrt{t})^{-2r+1} + (1+t-s)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}} \right].
\end{aligned}$$

□

Theorem 7.22 (Behavior). *Suppose $u(x, t)$ satisfies $u_t = Lu + O(|u|^2)$ and $|u_0| \leq E_0(1+|x|)^{-r}$, $r > 2$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. Set*

$$\bar{U}_* = \int_0^{\infty} \bar{U}(s) ds + \bar{U}_0,$$

Then $|\bar{U}_*| < \infty$ and for some sufficiently large $M'' > M' > M$,

(7.49)

$$\begin{aligned}
& |u(x, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)| \\
& \leq CE_0 \left[(1+t)^{-\frac{1}{2}} (1+|x-at|+\sqrt{t})^{-r+1} + (1+t)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}} (1+\ln(1+t)) \right].
\end{aligned}$$

Proof. Recalling $|\bar{U}(s)| = |N(y, s)|_{L^1(y)} \leq CE_0(1+s)^{-\frac{3}{2}}$, we have $|\bar{U}_*| < \infty$. Now we break $|u(x, t) - \bar{U}_* \bar{u}'(x) \bar{k}(x, t)|$ into three parts exactly the same as (7.23). By (7.44) and (7.11), first two terms are trivial.

$$(7.50) \quad |v(x, t)| + O(|v_x||\psi| + |\psi|^2) \leq CE_0(1+t)^{-\frac{1}{2}} \left[(1+|x-at|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \right].$$

Like (7.25), we break the last term into four parts. Since $|\bar{U}(s)| \leq CE_0(1+s)^{-\frac{3}{2}}$,

$$(7.51) \quad II \leq CE_0(1+t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M(1+t)}} \int_t^\infty (1+s)^{-\frac{3}{2}} ds \leq CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}.$$

By (7.48), we have

$$(7.52) \quad \begin{aligned} III &\leq CE_0 \int_0^t (1+t-s)^{-\frac{1}{2}} (1+s)^{-1} (1+|x-at| + \sqrt{t-s} + \sqrt{s})^{-2r+1} ds \\ &\quad + CE_0 \int_0^t (1+t-s)^{-1} (1+s)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} ds \\ &\leq CE_0(1+|x-at| + \sqrt{t})^{-2r+1} \int_0^t (1+t-s)^{-\frac{1}{2}} (1+s)^{-1} ds \\ &\quad + CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \left[\int_0^{t/2} (1+s)^{-1} ds + \int_{t/2}^t (1+t-s)^{-1} ds \right] \\ &\leq CE_0(1+|x-at| + \sqrt{t})^{-r+1} (1+\sqrt{t})^{-r} (1+t)^{\frac{1}{2}} + CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \ln(1+t) \\ &\leq CE_0 \left[(1+t)^{-\frac{1}{2}} (1+|x-at| + \sqrt{t})^{-r+1} + (1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \ln(1+t) \right]. \end{aligned}$$

Since $|\bar{U}(s)| \leq CE_0(1+s)^{-\frac{3}{2}}$, the estimate of IV is exactly the same as (7.40) which is

$$(7.53) \quad IV \leq CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}.$$

By (7.47) and (7.50)–(7.53), we obtain the result (7.49). □

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